A DIAGNOSTIC STUDY OF THE TEACHING PROBLEMS IN HIGH-SCHOOL MATHEMATICS

WILLIAM DAVID REEVE

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A DIAGNOSTIC STUDY OF THE TEACHING PROBLEMS IN HIGH-SCHOOL MATHEMATICS

BY

WILLIAM DAVID REEVE, Ph.D.
ASSOCIATE PROFESSOR OF MATHEMATICS, TEACHERS COLLEGE
COLUMBIA UNIVERSITY

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A DIAGNOSTIC STUDY OF THE TEACHING PROBLEMS IN HIGH-SCHOOL MATHEMATICS

CHAPTER I

ORIGIN AND PURPOSE OF THE STUDY

1. Origin of the Study

For some time, changes in the content and method of teaching high-school mathematics have been advocated both by teachers of the subject and by others who are interested primarily in the welfare of the science or in the welfare of the pupils who study it.[1] In support of these proposals it is generally claimed that mathematics as traditionally taught does not meet modern needs,[2] that in years past the theory of mental discipline and the need of preparation for college have been the compelling motives, and that recent social and economic changes and the increase in the number of high-school pupils have greatly altered the present situation.[3]

Some of the proposed changes have been merely suggestions relating to omissions or additions in certain parts of the science, but in other cases they have been more radical and have amounted to a complete reorganization.[4] Thus, we find the recent National Committee on Mathematical Requirements recommending [5] that

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[1] S. C. Parker, Methods of Teaching in High Schools, chap. iv. Ginn and Company, 1915.

[2] Abraham Flexner, A Modern School. General Education Board, New York.

[3] David Snedden, Problems of Secondary Education, chap. xviii. Houghton Mifflin Company, 1917.

[4] H. C. Morrison, "Reconstructed Mathematics," in Thirteenth Yearbook of the National Society for the Study of Education (February, 1914), pp. 9–31.

[5] The Reorganization of Mathematics in Secondary Education, being the final report of the National Committee on Mathematical Requirements, p. 14. Dartmouth Press, 1923.

"in the junior high school, comprising grades seven, eight, and nine, the course for these three years should be planned as a unit with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years." Scientific workers also have pointed out that mathematics has not done what has been claimed for it, and they have made their recommendations accordingly.[1] Even an association of mathematics teachers has recommended "a one-year course in elementary algebra and geometry of a concrete sort, designed so far as possible to test the pupil's qualification for future mathematical study."[2]

In a few instances in this country attempts have been made during the last twenty years to break away from the traditional methods of teaching algebra and geometry.[3] Some of these attempts have sought to improve the content of algebra and geometry and the methods of teaching; while others have resulted in the organization of courses in unified mathematics, correlated mathematics,[4] general mathematics, and the like.

It is a well-known fact that the percentage of failures in algebra and geometry is too high in many schools.[5] For example, it is not uncommon to find as high as 25 per cent or 30 per cent of a normal group of pupils failing in freshman algebra, — a situation which cannot be justified from any modern point of view. Such high percentages indicate either that the work is too hard for the pupils at the lower levels of ability or that the teaching of the subject has not been entirely satisfactory.

Professor Thorndike has recently pointed out[6] that the secondary-school population of today is very different from what it was thirty years ago, "not only in their experiences and interests, but also in their inborn abilities." According to a recent report of the National Education Association[1] the number of pupils enrolled in high schools in this country had increased from 202,963 in 1890 to 2,229,407 in 1922. In this report we read that "if the population of the United States had increased as rapidly as its high-school enrollment since 1890, its general population would now be 687,861,591." That is, not only is the high-school population very different in nature from that of thirty years ago, but it is also much larger. Nevertheless, there is little concrete evidence that teachers and textbook-writers in general have recognized and appreciated these facts.

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[1] A. A. Michelson, "The Purpose and Organization of Physics Teaching," *School Science and Mathematics* (January, 1909), Vol. IX, pp. 3–4.

[2] Association of Teachers of Mathematics in New England, Preliminary Report on the Status of Mathematics in Secondary Schools (December, 1911), p. 11.

[3] H. O. Rugg and J. R. Clark, Scientific Method in the Reconstruction of Ninth-Grade Mathematics. George Banta Publishing Co., 1918.

[4] E. R. Breslich, "Correlation of Mathematical Subjects," *School Science and Mathematics* (February, 1920), Vol. XX, pp. 125–134.

[5] C. H. Judd, Psychology of High-School Subjects, p. 18. Ginn and Company, 1915.

[6] E. L. Thorndike and others, Psychology of Algebra, p. 2. The Macmillan Company, 1923.

All the factors mentioned above have complicated the problem of selecting the course in mathematics that best meets the needs of the modern high-school group. In recent years, however, certain movements have helped greatly in transforming mere study of the situation into genuine action. The most important of these influences have been the junior-high-school movement, the work done by certain laboratory and experimental schools, the College Entrance Examination Board, the report of the National Committee on the Reorganization of Mathematics in Secondary Education, departments of educational research in various institutions of learning, and the testing movement in general.

As a result of the conviction that the traditional courses in algebra and geometry were not meeting the needs of modern high-school pupils, I began in 1916, in collaboration with Professor Raleigh Schorling, to work out a course of study more in line with modern demands. Since the majority of high schools in this country were organized on the senior-high-school plan, which covers the school years from nine to twelve inclusive, it seemed best to begin our study in the ninth grade. Our experience in previous years at the University of Chicago High School in seeking to determine the subject matter best suited to high-school pupils led us to decide upon a course in general mathematics as the one most likely to realize the tentative objectives which we had in mind.

At first the subject matter of the course was given to the pupils topic by topic, by means of mimeographed sheets, but after considerable revision it was finally published in book form.[1] The experimental work was carried on mainly at the University of Minnesota High School in Minneapolis, although a number of mathematics teachers in Minneapolis and St. Paul, who were interested in the experiment and who favored the general-mathematics idea, assisted us greatly by using the mimeographed sheets in their classes and by giving the tests which furnish the basis for the results described in subsequent chapters. The teachers who assisted in the study are all in schools which are accredited by the North Central Association of Colleges and Secondary Schools, and the number of pupils represented in the results of the various tests can be found by consulting the table on page 102.

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[1] *Journal of the National Education Association* (November, 1924), Vol. XIII, p. 271.

In seeking to develop a desirable course of study it was found that at least four things must be kept constantly in mind: (1) the aims or objectives to be realized in the course, (2) the content best suited to realize these aims most fully, (3) the best methods of teaching the content, and (4) a critical study (which follows as a corollary to the third item) of the most economical and useful ways for pupils to learn. In order to check the work by these four considerations it was necessary to establish an elaborate system of testing which has extended over a period covering the last eight years.

2. Purpose of the Study

It is not the purpose of this study to discuss the nature or the appropriateness of the objectives which governed the selection of the material in the general-mathematics course referred to above, or to attempt to justify the material as being of the type best fitted to realize these objectives in ninth-grade classes, although I believe in the value of such a course for many pupils. The purpose of this study is to encourage a better use of tests in mathematics, and particularly the kinds of tests which can be used by teachers to improve their instruction. Such tests, according to Professor Woody,[2] "involve almost all aspects of administration and of teaching. They are the best available instruments of appraisal of school work and are an essential part of the program of every progressive school system."

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[1] R. Schorling and W. D. Reeve, General Mathematics, Book One. Ginn and Company, 1919.

[2] C. Woody, "Standard Tests and their Use — A Symposium," *Teachers College Record* (October, 1924), Vol. XXVI, p. 101.

For a list of modern objectives the reader is referred to pages 6–10 of the report of the National Committee previously mentioned. This list should furnish a point of departure for any teacher or group of teachers who may wish to formulate a list of fundamental objectives. For those who desire to read further in connection with the objectives to be attained in mathematics the bibliography on objectives on page 115 will be found helpful. Any reader who is particularly interested in general mathematics will find references in the bibliography beginning on page 114.

3. GUIDING PRINCIPLES IN MAKING TESTS

I have tried to keep in mind the following guiding principles which should be used in constructing tests:

First, all tests should cover only subject matter that has been presented to the pupils. If pupils who have the native ability to learn a certain thing fail to do so, there is a chance for remedial work to bear fruit. On the other hand, there is little to be expected or gained by remedial work in the case of pupils who have learned as well as can be expected in relation to their innate ability. If both the diagnosis and the subsequent remedial work suggested by the results of the tests are to be of value, the teacher must have at hand specific measurement of the pupil's achievement, which can be fairly made only in a field in which he has been working.

In improving instruction, tests that contain material which has not been previously taught cannot have much diagnostic value. For example, in a class where many pupils are not familiar with the subject matter, such tests cannot be used to determine whether or not the content of the course was too difficult or whether or not the teaching was well done. This point will be further considered in a later chapter.

Second, the tests should emphasize those parts of the subject matter which are fundamental and to which the pupils have directed the most attention. Nothing should receive attention in the tests that is not worth perpetuating in the course. Indeed, certain recent developments lead me to feel that some of the tests mentioned in this discussion do not conform to this second guiding principle as well as they should.

Third, the scoring of the tests should be as objective as possible instead of being dominated by the subjective methods which have characterized our measuring in the past. It has been my purpose to construct both the tests and the method of scoring so that different teachers, employing them in measuring the same abilities, may obtain practically the same results.

The attempt, therefore, has been made to devise tests that measure adequately the pupil's knowledge of the subject matter studied by the majority of the class, to assist the teacher in selecting only important topics, to stimulate the pupil to greater effort, and to aid him in self-instruction.

It should be clearly understood that the method of using these tests is not at all dependent upon the type of subject matter taught in the course. The fact that a general-mathematics course was used as a basis for the tests is only incidental to the use that can be made of them. If a teacher believes in a course of this type, the tests described in subsequent chapters will be all the more interesting and helpful. If the teacher does not approve of such a course for the ninth grade, the method of testing herein described can still be used in building up a series of tests to fit the particular type of course that is preferred.

The most common types of mathematics tests now used in this country are discussed in the second chapter, with particular reference to the advantages and disadvantages of each type and their significance in modern education.

For a more elaborate discussion of the history and significance of the general testing movement the reader is referred to the bibliography given on page 116.

CHAPTER II

MODERN TESTS IN MATHEMATICS AND THEIR SIGNIFICANCE

1. PROGNOSTIC TESTS

Tests in mathematics may be classified under two main heads: namely, prognostic tests and achievement tests. Prognostic tests are those which are given early in the pupil's career, for the purpose of measuring his innate ability to do mathematical work. By their use the teacher is enabled to predict the probable success of a pupil in his later work. For example, Professor Rogers's prognostic tests[1] were designed to measure the specific ability of pupils to succeed in studying secondary mathematics.

Prognostic tests were developed to meet the need for some kind of instrument to measure a pupil's capacity to learn mathematics, in order that he might be more intelligently advised with respect to his subsequent work. Since previous tests of this type, except those relating to geometry, dealt only with the more mechanical phases of the work, mathematicians objected to them on the ground that there is little in common between the habits of symbol manipulation, which was emphasized by nearly all the early tests, and the more fundamental processes of reasoning which are characteristic of higher mathematical study. In general they maintained that the intensive tests in special fields like arithmetic, algebra, and geometry failed of their purpose for the reason that these subjects are rarely applied by themselves. It should be observed, however, that everything depends upon what purpose the maker of a test intends it to serve. If it is not intended to be prognostic, there may be justification for such a test on some other basis.

No one seems to have claimed that prognostic tests can be, or should be, the sole basis for prediction. Nevertheless, they

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[1] A. L. Rogers, "Tests of Mathematical Ability and their Prognostic Value," *Teachers College Contributions to Education*, No. 89.

constitute an important step in the direction of trying to discover those pupils who give the most promise in the field of mathematics. If such tests can be of assistance in helping us in our problem of discovering individual differences in mathematical ability, they may become useful instruments for prognosis in our schools. It should be insisted, however, that they must not be made the sole basis for prediction, classification, or guidance, because such absolute dependence would overlook other important factors. Professor McCall says:[1] "There are significant segments of each pupil's make-up which tests do not now touch. In occasional instances this segment has more importance for classification than the segment measured by tests. Teachers' judgments appear to be the only immediate hope of measuring pupils' purposes. Furthermore, the teacher can weight physical, emotional, and social characteristics in ways that tests cannot." Speaking of Professor Rogers's prognostic tests, Professor Thorndike says:[2] "Dr. Rogers's tests are the best so far published to select according to promise of ability in algebra and geometry. We have evidence, however, that their value consists chiefly in their being a good measure of abstract intellect, and that any reliable measure of abstract intellect would prophesy success in algebra and geometry nearly as well. Also it seems probable that algebraic ability and geometrical ability differ nearly if not quite as much as do ability in algebra and ability in any other abstract subject such as physics or Latin. Consequently, tests specialized for numerical and spatial data may be found to do this prognostic work even better than the Rogers tests.

"If these tests could be arranged so as to be given before any algebra or geometry had been studied, it would, of course, be an added advantage. The Rogers tests presuppose a few months' study of algebra, and are much influenced by familiarity with the idea of and procedure in geometrical proofs.

"It seems probable that such a prognostic test for algebra could be made by using the Rogers tests 3 and 5 with a test in arithmetical problems and a test in disarranged numerical equations."

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[1] W. A. McCall, How to Measure in Education, p. 59. The Macmillan Company, 1922.

[2] E. L. Thorndike and others, The Psychology of Algebra, pp. 216–217. The Macmillan Company, 1923.

In a recent article[1] Professor David Eugene Smith says: "The prognostic test at its best achieves quickly and with improved results that which the schools have heretofore discovered after a loss of valuable time; at its worst it leads into a determinism that is more dangerous than the extreme form of Calvinism which left each individual absolutely without hope. On the whole the tests have achieved a great and well-deserved success, and this success will be much more apparent when a new generation comes forward to correct the errors of the present one."

If such tests can be made a reliable help in prognosis, we should all favor a wider use of them in selecting those pupils who are most likely to profit by advanced work in mathematics. We might, and probably should, then require mathematics only through the ninth year (the third year of the junior high school), and should limit the subsequent courses to those who are capable of a much higher grade of work than is now possible. Such prognostic tests would then be the means of doing justice to a large number of pupils who are forced to study mathematics against their will and in many cases without much gain, or at least they could be used to advantage in determining what kind of mathematics would be most valuable to a particular pupil. If we had in the tenth-year, eleventh-year, and twelfth-year classes in mathematics only those pupils who liked the subject and were able to learn it, the joy that would come to teachers and pupils alike would more than offset the loss in numbers that would result from the adoption of such a program.

2. ACHIEVEMENT TESTS

While prognostic tests are concerned with finding out what a pupil is able to learn, it is the function of achievement tests to determine what he has learned. Such tests are intended to measure the progress which a pupil or a class is making in a given topic or course. They also measure indirectly the pupil's innate ability, since whatever progress he makes is necessarily dependent upon this ability. These tests, however, are so affected by classroom influences that they cannot be accepted as reliable instruments for the accurate measurement of capacity to learn, although in general they correlate rather highly with intelligence. Professor Monroe divides achievement tests into two types, saying:[1] "Since ability tends to be specific, we may recognize two types of achievement tests: first, those which measure separately certain specific achievements; and, second, those which yield average or general measures of a number of specific achievements within a given field. A test which yields separate measures of specific abilities is frequently called a diagnostic test, because it points out or diagnoses the specific weaknesses of pupils." As a matter of fact, however, both prognostic and achievement tests may rightly be considered diagnostic in nature, although in this country we have thus far made little use of diagnosis in prognostic tests. Furthermore, any achievement test may be made diagnostic even though it may not be so intended, and an achievement test may have diagnostic value and still not be used for purposes of diagnosis.

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[1] D. E. Smith, "On Improving Algebra Tests," *Teachers College Record* (March, 1923), Vol. XXIV, pp. 87–88.

It is our purpose at this time to emphasize the importance of diagnostic tests as instruments for aiding the teacher in improving instruction. As stated on page 5, the main purpose of diagnosis is to guide the teacher in remedial instruction. Without some plan of discovering the particular defects of a pupil or of a class the work of the teacher is likely to be more or less futile.

If the proper kinds of tests are given, the progress made by pupils can be measured, and the norms of accomplishment thus established can be used by the teacher as a guide in discovering the educational needs of future classes. In addition to this the pupils can assist the teacher greatly by locating not only their particular difficulties but the causes of these difficulties as well. The assistance rendered in diagnosis by the pupils' introspection should not be overlooked, because a pupil will often find the cause of a difficulty more quickly than the teacher.

3. TYPES OF ACHIEVEMENT TESTS

Two types of achievement tests have had rather wide use in this country. The first type includes the tests set by the teacher alone or in coöperation with others who are more or less responsible for the existing course of study; the second includes the "extramural" tests set by examiners who have little or no direct contact with the classroom. Illustrations of the latter type are the ordinary College Entrance Board examinations, certain well-known state examinations, and the so-called standardized tests.

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[1] W. S. Monroe, The Theory of Educational Measurements, p. 40. Houghton Mifflin Company, 1923.

It is not the intention to minimize the importance of any one of the tests mentioned above, especially if they can be made to serve some useful purpose, but rather to emphasize a wider use of some types and a more intelligent use of others. In this connection Professor Wood says:[1] "Teaching is one thing, measuring another; when we wish to teach, let us not assume that a device which has pedagogical value also has measurement value, or that a device which has little or no pedagogical value has no measurement value. . . . Intrinsic pedagogical value in an examination is highly desirable, but the value of the examination as a measuring device cannot be made to depend upon its value as a teaching device."

4. TESTS SET BY THE TEACHER

An entire volume could be written on the subject of tests set by the teacher, but space will not permit any more than a statement of a few of their advantages and disadvantages. It is well known that some teachers are not qualified to make the careful analysis of the course that is necessary to determine the abilities that are fundamental and those that can or should be measured. As a result such teachers include in important tests a meager sampling of the large range of abilities, and regard these few questions as an instrument for measuring the entire field. In fact, too many teachers fall short when it comes to having a definite list of the significant things to be measured in a given course.

In the second place, with many teachers the marking of test papers is highly subjective; that is, the mark given depends to a large extent upon the person who does the marking. Furthermore, the measures obtained by the teacher from many of the ordinary essay examinations are not truly diagnostic; that is, they do not point out the particular details upon which the pupil is either weak or strong. In fact, the present marking systems of both high-school and college teachers are usually subjective and artificial. Professor Wood brings out some of the important factors in a very illuminating way when he says:[1] "The essay examination in the hands of the average teacher does measure a very important element which apparently cannot be measured directly by any other means thus far developed. But it measures that element very incompletely and very unreliably." It is thus clear that if specific measures for diagnosis are desired, the ordinary examination will not prove entirely satisfactory. Instead of continuing the practice of doing injustice to thousands of pupils every year because our tests are misused, we must in some way or other develop a new marking system which will enable us to rank their achievement in relation to their ability.

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[1] B. D. Wood, Measurement in Higher Education, p. 174. World Book Company, 1923.

In spite of the frequent inadequacy and inaccuracy of teachers' judgments both in setting good examinations and in marking them fairly, it should be more generally realized that these same teachers are in the long run the ones best situated to do the task. Teachers can not only learn how to make objective tests that will have both measuring and diagnostic value, but they can also learn to use them intelligently. This ability to use the tests will increase in proportion to the progress the teachers make in understanding scientific methods of measurement. Instead of constantly reminding teachers of their failures, without making suggestions for improvement, attention should be directed to the traditional method of marking rather than to the teacher who, because of the lack of a better method, is forced to use one that is neither exact nor reliable. School marks have long been used to measure a multitude of things which have not and cannot be measured with any degree of precision. By a proper development and use of objective tests we may be able to work out a better marking system for the future.

5. TESTS SET BY OTHERS THAN THE TEACHER

During the past generation there have been many discussions regarding the examinations set by state and college authorities. Some progress has been made in the content of these examinations and in the method of conducting them, but much remains to be done before they can be said to have reached their highest development even for the purpose for which they are designed. Speaking of such tests, Professor David Eugene Smith says:[1] "New York State has dictated regents examinations, generally good for a poor teacher and generally bad for a good one. The College Entrance Examination Board has also, and naturally, dictated what should be taught in algebra, and has recently made a long step in advance by a series of improvements. Each of these cases of dictation has contributed powerfully to making algebra stagnant, and each has been potent in keeping it on a dead level of traditional mechanism." Professor Smith has implied in the preceding statement that the people who formulate the tests referred to above are often better able to judge what is fundamental in a course than are some classroom teachers of mathematics. In fact, the recent College Entrance Examination Board syllabi reflect more modern types of mathematics courses than those which a great many teachers are now using. The trouble is not so much with the original purpose of such examinations as with their imperfect use and interpretation. In many cases these examinations measure abilities which have not been considered in the previous instruction of many of the teachers whose pupils have to take the examinations, and consequently it follows that the results secured cannot be called fair measures of the teachers' effectiveness. Nevertheless it is true that the efficiency of teachers and the efficiency of classes have been compared on the basis of the scores made on such examinations. As a result all sorts of methods are resorted to by unscrupulous teachers whose sole aim is to get their pupils through these examinations with a passing grade. Such practices, it need hardly be said, do not represent a desirable type of modern education. Since such examinations, as now regulated and administered, have not given satisfactory results, the question arises as to whether or not it would be a wise plan to replace them with intelligence tests or prognostic tests of some kind that can be used as a basis for determining what pupils are able to carry on higher mathematical work.

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[1] B. D. Wood, Measurement in Higher Education, p. 161. World Book Company, 1923.

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[1] David Eugene Smith, "On Improving Algebra Tests," *Teachers College Record* (March, 1923), Vol. XXIV, p. 87.

Using two examinations that were set by the College Entrance Examination Board in June, 1921, — one in algebra, known as "Mathematics A," and one in geometry, known as "Mathematics C," — Professor Wood finds that "the reliability of the examination itself in algebra is unsatisfactory, and in geometry entirely unacceptable."[1] The College Entrance Board, however, evidently does not agree with Professor Wood, as can be seen by examining some of its recent reports.[2]

That teachers of mathematics are not satisfied with College Entrance Board examinations in mathematics is also evidenced by a recent report of the Committee on Examinations of the Association of Mathematics Teachers of New Jersey.[3] In this report it is charged that the present College Entrance Board examinations in mathematics do not give the largest possible measure of specific preparation, — first, because the form of the examinations is at fault as a measuring instrument; second, because the topics chosen are not weighted in accordance with their value as a preparation for advanced study; and, finally, because the present "comprehensive" examinations are not what their name suggests; that is, they are not composites of examinations which, taken individually, seek to test important topics or abilities.

It is evident, then, that these examinations do not meet the requirements for tests outlined on pages 5 and 6. They tend to measure less of achievement and more of intelligence. Furthermore, such examinations cannot be given to all pupils except at the expense of certain groups. In fact, it is easy to conceive a situation in which one group whose preparation had been of a higher type than that of another would make a showing distinctly inferior to that of the second group simply because the examination was not based upon material to which the first group had devoted an equal amount of attention.

Since the College Entrance Board examinations, state Regents' examinations, and the like are not devices which measure adequately a pupil's ability, why should they not be given their proper place and weight in our educational scheme? Better still, as suggested above, if their only purpose is to secure the most capable pupils for college work, why cannot examining boards be induced to use other methods[1] instead of the traditional examinations? If that were done, all high-school pupils could be given that type of education in mathematics which would be most useful to them, whether they went to college or not.

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[1] B. D. Wood, The Reliability and Difficulty of the College Entrance Examination Board Examinations in Algebra and in Geometry, p. 14. Published by the Board, 1921.

[2] Reports of the Secretary of the College Entrance Examination Board for 1921, pp. 1–4; also for 1922, pp. 18–19, and for 1923, pp. 1–3 and 7–9.

[3] Preliminary report, *The Mathematics Teacher* (October, 1923), p. 374.

6. STANDARDIZED TESTS

The so-called standardized tests have recently come into rather wide use in this country, but their introduction has been accompanied, in a great many places, by grave misuses of the tests. Sometimes those in charge of the testing program are so unwise as to judge the success or failure of a teacher solely by the outcome of certain tests. In some cases invidious comparisons made on the basis of test results have not been checked by a careful study of the methods of teaching employed. In still other cases certain standardized algebra tests have been regarded as entirely reliable, and it has been assumed that they could be used to measure every feature of the teaching of that particular subject.

Professor Thorndike says[2] that "the testing movement is embarrassed by its success." He says further that "improvement in the instruments of measurement, both of intellect and of school achievement, is more desirable than multiplication of their number." It is proper to add that in a transition period of curriculum construction, like the one in which we find ourselves, we should be particularly careful to seek only an intelligent use of all tests. Mr. Barber points out[3] that "it is vastly more important just now to get our mathematics redefined than it is to develop and adopt standardized tests which might serve to crystallize it and so retard or prevent the redefinition."

A great deal of the traditional algebra which has been questioned by many teachers and textbook-writers as contrary to modern objectives has been replaced by more valuable material. It is an interesting fact, however, that standardized tests have made difficult the general acceptance of some of the more modern algebra courses. The values claimed for standardized tests because of their carefully selected content have not always been legitimate. To quote Professor David Eugene Smith again:[1] "The complaint is not so much that the tests are solely mechanical, involving only a minimum of intellectual processes, — a fault that is probably inevitable in the present stage of development, but which is being successfully removed in some of the arithmetic tests; it is also that the material required for testing the mechanical processes is often such as should play only a minor rôle, if any, in the education of the average citizen. The tests represent generally a dead level of dull grind, offering to the teacher only this ideal of an algebra course. He may escape from the curriculum, making his own course; he may and should select from the textbook that which he needs for carrying out his plan; but he cannot escape from teaching those things that are required by outside examinations whether they be set by boards of regents, by colleges, or by educational testers." Professor Smith goes on to point out a large number of examples taken from recent standardized tests in algebra, some of which he commends, but many of which he condemns, and rightly so. For example, he quotes the following exercise from one test:

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[1] W. M. Proctor, "The High School's Interest in Methods of Selecting Students for College Admission," *School and Society* (October 10, 1925), Vol. XXII, pp. 441–448.

[2] E. L. Thorndike, "Standard Tests and their Use — A Symposium," *Teachers College Record* (October, 1924), Vol. XXVI, p. 93.

[3] H. C. Barber, Teaching Junior High School Mathematics, p. 134. Houghton Mifflin Company, 1924.

Divide $$\frac{\sqrt{2}}{\sqrt{3}} \text{ by } \frac{\sqrt{2}}{3}.$$

He then adds: "Why should anyone ever wish to perform this division? It is not algebraic, and it has no particular significance either in algebra or in connection with formulas that are likely to arise. It is possible to test our schools on something that we have some other reason to perpetuate than the mere reason for meeting this kind of test."

Then, as another puts it, "Too often tests are given, the data are tabulated, conclusions drawn therefrom are utilized by supervisors, and methods are revised by teachers because of them; but the pupils who wrote the tests are not informed of any of the results except in those rooms where unsatisfactory conditions have brought about attempts to shift the blame to these pupils." Here lies the trouble in the use of many of the standardized tests. Teachers and pupils alike are often given little consideration in their administration, and the result is that great loss ensues in the improvement of instruction.

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[1] D. E. Smith, "On Improving Algebra Tests," *Teachers College Record* (March, 1923), Vol. XXIV, p. 88.

Professor Woody[1] has pointed out that "from one point of view standardized measurement represents the refinement obtained through the crossing of current practice and scientific method." No doubt many of our test-makers have appreciated this point of view, but we might have made better use of some of their work if it had been more generally understood earlier.

Teachers always have measured and always will measure their pupils in some way or other, but the science of measurement will not reach its maximum of importance until the teachers and the makers of tests establish a partnership. Each may then hope to develop scientific methods of approach, and only then will the teachers themselves have an abiding interest in the outcome of their work. It is not essential that teachers should construct tests themselves, although many will soon learn the technique of the process; but they should at least be consulted in order to assure a result that shall be generally useful to all concerned. It may be, as Professor Trabue has stated,[2] that "college professors and graduate students of education may continue to build and improve the measuring instruments used in testing school pupils; but the diagnosis of specific educational diseases and the long experimentation necessary to discover whether for a certain type of pupil more drill on forms will be helpful or fatal, — the 'case work' which constitutes the next step in the use of educational measurements, — must be done in the classroom by the regular teachers."

A careful study of many of the standardized tests shows that they have been prepared by people who select material with little regard to the proper objectives to be obtained and who seem to be ignorant of the plans for the reorganization of mathematics. Statistically the tests themselves may be perfect; but when a task is introduced into a test merely because it represents a certain degree of difficulty, it simply tends to prejudice all persons of common sense against the whole movement. It is such unfortunate errors as these that retard progress.

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[1] C. Woody, "Standard Tests and their Use — A Symposium," *Teachers College Record* (October, 1924), Vol. XXVI, p. 95.

[2] M. R. Trabue, "Standard Tests and their Use — A Symposium," *Teachers College Record* (October, 1924), Vol. XXVI, p. 111.

It is not difficult to formulate satisfactory standardized tests that are more objective than the older types, that are more uniform with respect to administration and the scoring of results, that furnish desirable norms of achievement for guidance in future instruction, and that are much more reliable as measuring instruments. It is not certain, however, that we should attempt to emphasize the standardization of tests as measuring devices before we know what abilities we wish to measure. It would seem that at the present time we should be more interested in determining clearly the purposes in view in the teaching of mathematics, the content best fitted to help us realize these purposes, and the kind of tests that will afford a check upon our results. This does not mean that no measuring should be done in the meantime, but rather that our methods of measuring should be improved before we seek to increase the use of standardized tests.

In all fairness to standardized tests, however, it should be said that they have gone beyond what Professor Woody calls the "curiosity" stage and "the stage in which the predominant idea was the use of the tests for determining existing levels of achievement," and, in some respects at least, have approached the third stage, "in which the predominant idea is the utilization of tests as a means for the improvement of instruction." They have been helpful in this third stage, however, only as they furnish facts concerning certain "levels of efficiency" reached by pupils, and thus "contribute to the evaluation and diagnosis of the efficiency of instruction."

Where standardized tests are valid and reliable instruments, they may be profitably used for purposes of "general-survey diagnosis," and even in some cases for class and individual diagnosis; but this work must be based more and more on the coöperation of all concerned, from the superintendent down to the pupils themselves.

The indications are that we can confidently look forward to a time not far distant when an organization like the National Council of Teachers of Mathematics, with the help and advice of all concerned, will be able to develop something like national standardized tests in mathematics. These tests can be made in such forms that they may be used not only for both general and particular diagnosis but also as aids in improving instruction. When this is done, the establishment of norms of attainment which can be generally used will follow. We should realize, however, that a teacher is not always justified in feeling satisfied with the results when pupils reach a certain norm of achievement. To do this is to overlook the fact that the standard norms may be raised by lifting the general level of achievement through better methods of teaching. On the other hand, "they need to see that the law of diminishing returns applies to educational products as well as to economic products, and that continually trying to raise the level of achievement may result in educational bankruptcy."

Professor Monroe thinks[1] that "in considering the need for standardized objective tests which shall be uniformly used by all teachers, it is necessary to bear in mind the possibility that the measures obtained by the crude methods may be as effective in some, or even all, of these activities as the measures obtained by standardized objective tests. Thus, when we show a need for measurement of abilities of pupils, it does not necessarily follow that we have established a need for standardized objective tests." Furthermore, he says:[2] "The fact that much of the content of the subjects studied in the seventh and eighth grades and in the high school is not fixed, makes it impossible to construct satisfactory instruments for measuring the achievement of pupils. A test which would measure the outcomes recognized by some teachers as minimum essentials would be criticized by other teachers as failing to measure other outcomes which they considered essential. Until there is a greater degree of agreement concerning the minimum essentials of the subject taught in these grades, it will not be possible to construct standardized tests which can be recommended for general use in diagnosing pupils with respect to their achievements. This condition does not remove the need for diagnosis, but it should be made by instruments and methods which are adapted to the instruction which the pupils have received." It is this last point, in substance, that I had in mind when I began to develop the new type of test which is described in the next chapter.

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[1] W. S. Monroe, The Theory of Educational Measurements, p. 42. Houghton Mifflin Company, 1923.

[2] Ibid. p. 49.

It is conceivable that we might realize our educational objectives without a national syllabus in mathematics; indeed, it might be better to do without one. It is not conceivable, however, that we shall be able to succeed separately unless we develop, for general mass instruction, a type of teaching that is based upon the specific needs of individual classes and pupils. This sort of instruction should be remedial and must be based upon specific diagnostic measures of pupils' achievements.

CHAPTER III

THE SELECTION OF MATERIAL FOR THE TESTS

1. The Content of the Course

In addition to basing our reorganization of the course in mathematics upon the general-mathematics idea, Professor Schorling and I omitted a great deal of traditional material and organized the rest in what we believed to be a more psychological way. We attempted to develop a better and more modern method of approach. We took for granted many geometric ideas which pupils have succeeded in proving only after a considerable number of basic theorems have been developed. For example, the work was planned so that with a few basic definitions and theorems it was possible to prove the Pythagorean Theorem, while such cases as "a perpendicular is the shortest line from a point to another line" were taken for granted.

The course was developed around the following topics: (1) the equation; (2) linear measurement; the equation applied to length; (3) properties of angles; (4) the equation applied to area; (5) the equation applied to volume; (6) the equation applied to fundamental angle relations; (7) the equation applied to the triangle; (8) positive and negative numbers; addition and subtraction; (9) positive and negative numbers; multiplication and division; factoring; (10) graphical representation of statistics; the graph of a linear equation; (11) the formula; graphical interpretation of formulas; (12) functions; (13) similarity, construction of similar triangles; (14) indirect measurements; scale drawings; trigonometry; (15) theory and application of simultaneous linear equations; (16) geometric and algebraic treatment of roots and powers; (17) logarithms; (18) the slide rule; (19) quadratic functions and equations.

Tests were then devised for measuring the success that pupils achieved with the various topics.

2. THE EXPERIMENTAL WORK

In developing this course of study an effort was made to introduce problems of an informational type concerning important public buildings in various parts of the country, and problems about the areas of certain bodies of water, the lengths of well-known rivers, the heights of certain mountains, and the like. This type may be illustrated by the following problem taken from a recent text on algebra:

"The elevation of Mt. Whitney in California, the highest point recorded in the United States, is 14,501 ft., measured from sea level. The lowest point of dry land in the United States is in Death Valley, California. If 52 times the elevation of Death Valley be diminished by 45, and the result be increased by the elevation of Mt. Whitney, the sum is zero. Find and interpret the elevation of Death Valley."

The artificiality of this problem is evident, and the same may be said of similar problems found in many of our modern algebra texts.

The results of giving such problems to our pupils to solve were very unsatisfactory. From the pupils' own introspection, aided by such questions as would naturally be raised in trying to determine the pupils' difficulties, it was discovered, in the first place, that the problems were not interesting to the pupils; secondly, that they were not real to them in the best sense of the term; and, finally, that they were too hard for them to read. When we remember that it is important in teaching to consider only one major difficulty at a time, it is not hard to see why such problems cause trouble when the pupil is getting his first experience in solving simple equations in one unknown.

At first sight it may seem strange that the same pupils who found trouble in obtaining the correct equation to represent the conditions in the type of problem quoted above could obtain the result quite easily when a simple formal equation was given. The same thing appeared to be true with many of the applied problems,— the so-called real problems commonly found in current textbooks. For this reason we endeavored in our experiment to omit such problems, with the exception of a few of the less objectionable ones which might be used to challenge the thought of the more brilliant pupils. While it may be argued that all such problems should be left in the course for the same reason, it is doubtful if this is justifiable.

Our experience in the course of the experiment convinced us that a problem is not real to a pupil merely because it seems so to the teacher; nor is it real because the problem is taken from physics, chemistry, or any of the other sciences, which later prove decidedly interesting to certain pupils. Many of my own classes have considered puzzle problems of one type or another more real than some of the problems from the laboratory that I thought would interest them, and it is not surprising that they should. That is why problems about the hands of a clock can often be used with profit in a course in algebra, while some of the real problems arouse but little enthusiasm. This entire field of pupils' interests and the influence which interest should have on the successful solution of problems need to be fully explored. According to Professor McCall,[1] Nature has an "excellent device for protecting pupils. When the pupil is presented with a problem beyond his capacity Nature automatically cuts the circuit and nothing happens."

Speaking of the relative values of problems, Professor Carson, in a very interesting discussion on "The Useful and the Real," [2] remarks that pupils' "experiences and imaginations run in grooves more or less alike; they [the pupils] are interested in puzzles, hidden treasures, travels, railways, ships, and the like, and problems concerned with these entities are real to them no matter how absurd they appear from the standpoint of practical life. Their educational utility is not to be measured by their commercial or scientific value, but by their degree of reality for the pupils under instruction." Moreover, he says,[3] "There is no doubt that the emptying of cisterns, the coincidence of clock hands, and other seeming trivialities do exemplify the handling of rates in ways which are more real to young children than others which have more actual utility, and they are therefore to be welcomed rather than condemned. The mistake in their treatment, and as gross a mistake as could well be made, has been their grouping by subject matter instead of principle. All questions which deal with one principle should be grouped together and the subject matter varied continually."

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[1] W. A. McCall, How to Measure in Education, p. 85. The Macmillan Company, 1922.

[2] G. St. L. Carson, Mathematical Education, p. 42. Ginn and Company, 1913.

[3] Ibid. p. 43.

In the course of our experiments we found that it was best, as far as possible, to have the definition follow as the outgrowth of development work rather than to give it as the basis for such work, and to let "principles be the products of habits rather than their producers." Time after time we were forcibly impressed by the need of presenting "one major difficulty at a time" wherever this was possible. For example, in our tentative course we had planned to teach pupils how to solve an equation like $\frac{3}{4}x - 5 = \frac{2}{3}x - 4$ before they had been taught to solve an equation in which the unknown appeared in both members. They had been taught to solve an equation containing fractions; but having both the fractions and the unknown on the right side at the same time was too difficult for them. It is clear that we were not justified in expecting them to understand this type without adequate preparation. The same error of judgment often characterizes the work of other teachers. With our greater maturity we all tend to expect our pupils to hurdle difficulties that to them are well-nigh impossible. As a result of our experience it was decided in the course of the experiment first to teach the pupils how to solve equations like $3x + 4 = 2x + 9$, in which the unknown appeared on both sides but without fractions, and then to introduce the fractions later in the same equation. This plan at once produced significantly better results, although naturally they were not perfect. It is probable that still better results could have been obtained, but under present conditions the methods of conducting recitations make the complete mastery of any topic impossible for all pupils, even if it were desirable.

It was also found that pupils could solve certain types of problems successfully and with interest, although, judged according to modern aims in teaching, the policy of including these problems in the course was questionable. How important is the fact that pupils like to solve certain types of problems with which, as a rule, they are successful? For example, pupils enjoy solving equations in two unknowns partly because they are fairly successful in this work. They are not, on the whole, as successful with much of the work connected with simple and important formulas. What methods shall we use to obtain a list of formal and applied problems that shall be approximately ideal for the ninth grade? Here is another question that still awaits solution.

The question also arises as to what should be done with certain problems in algebra that for a long time have appeared to be too difficult. Are they to be discarded for this reason alone, or is it not possible that with better teaching and better methods of study these problems may prove to be satisfactory? If, through improved methods of presentation, they can be suited to the pupils' understanding, are they really worth the time it takes to teach them? These and many similar questions have been raised during the progress of this study. It cannot be claimed that any of them have been fully answered, but it is possible that the results secured in the course of the experiment may throw some light on many of these teaching problems. In any case it is hoped that the results will stimulate others to carry on the work where we were unable to find satisfactory answers. If this study should stimulate a real interest in diagnostic tests, a desire to obtain a better course in mathematics, and genuine attempts to find out how children learn, the time spent can then be justified.

3. PRACTICE TESTS

After the initial diagnostic work in our study had reached a stage of any importance in the testing program, timed practice tests were introduced as teaching devices. A considerable amount of time was lost in the earlier stages in trying to combine mass instruction and individual instruction in some usable fashion. It does not appear that the problem was solved in this experiment, nor do others seem to have been any more successful. One result, however, seems to have been reached: namely, that, of the two, individual instruction should receive the greater emphasis. It is here that the value of the practice tests appears in enabling the teacher to discover quickly the pupils who are in need of help, and to keep from wearying the brilliant pupils with work that they do not need. Furthermore, they help the pupil to measure his own progress in any given topic, and to be more intelligent in calling upon the teacher for assistance when this progress is not satisfactory. There is little danger that such tests will be overdone if they are used also for diagnostic purposes.

Professor McCall states the case succinctly when he says,[1] "Practice tests bring swift aid to the pupil who needs it, and prevent teaching when it is not needed." Overlearning is better than underlearning, but why should the teaching of anything be continued when it is no longer needed? Practice tests will help us to avoid past mistakes in this respect.

In the preliminary study of practice tests in this experiment only those items were chosen which were thought to be of fundamental importance in developing abilities. In the light of later findings, however, it appears that some of these items may well be omitted from any final course.

It was found, as usual, that some pupils did much better than others, but failure on the part of a pupil was not always his fault. Occasionally the teaching was not effective; in some cases certain exercises or problems were too difficult; and in other cases the material was so easy as to be solved by practically all pupils, a point which will be more fully discussed in Chapter IV. It was also found that sometimes the work was harder for pupils in one school than in another, although the pupils were relatively of the same age and ability. In general, however, the results were much the same.

A large number of timed practice tests, samples of which are shown on pages 27 and 28, were made for use in connection with the various topics. In the work on the equation two kinds of timed practice tests were used. The first was a series of exercises in which formal equations were to be solved; the second was composed of a number of applied problems in which either the answer or the equation expressing the conditions of the problem was required. In this way we tested the ability of the pupils to solve certain types of formal equations in one unknown, and also their ability to translate statements expressed in words into algebraic language.

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[1] W. A. McCall, How to Measure in Education, p. 118. The Macmillan Company, 1922.

The equation $x+3=6$ may be taken as a sample of the first type given in the course. In fact, this type was given in the first lesson of the year, which was devoted to teaching the pupils how to solve such an equation by subtracting the same number from each side. Positive and negative numbers were not introduced until much later in the course, as can be seen by referring to the outline of topics on page 21. Speed was encouraged in these practice tests, but accuracy was given a higher value. A business man need not be particularly rapid in his computations, but there must be no question as to his accuracy.

After two lessons on the above type of equation the following test and the accompanying directions were given, for the purpose of encouraging speed and developing accuracy in the fundamental operations:

TIMED PRACTICE TEST

Time, 2 min.

Without copying, solve the following equations:

1. $x+3=6$.
2. $x+2=7$.
3. $x+5=9$.
4. $x+7=12$.
5. $x+1=15$.
6. $x+13=23$.
7. $x+15=37$.
8. $x+30=50$.
9. $x+37=41$.
10. $x+17=33$.
11. $x+2.5=4.7$.
12. $x+3.9=7.9$.
13. $x+3.8=4.5$.
14. $x+4.3=8.1$.
15. $x+12.2=17.4$.
16. $x+19=77$.
17. $x+23=96$.
18. $x+48=101$.
19. $x+96=150$.
20. $x+72=212$.
21. $x+15.3=21.7$.
22. $x+16.9=33.5$.

The median number of rights on this test is 15.

To determine the median number of "rights" (exercises correctly solved) in any class, find the score (the number of rights) which has as many above as below it. It is not expected that many, if any, pupils in an ordinary class will complete all the exercises correctly in the time allotted for any test.

TIMED PRACTICE TEST

Time, 5 min.

Write the answer to each problem, but do not solve any equation:

1. What is the cost of 8 oranges at c cents each?

2. A man had m dollars and lost n dollars. How many dollars had he left?

3. If a boy is n years old now, how old will he be x years from now?

4. Three times a certain number decreased by 4 is 20. Find the number.

5. Find the number of feet of wire needed to inclose a lot l feet long and w feet wide.

6. If 4 is subtracted from three times a certain number and 8 is added to twice the number, the results are equal. Find the number.

7. Find two consecutive numbers whose sum is 47.

8. Find two consecutive odd numbers whose sum is 76.

9. One man has three times as much money as another man. If both together have $5400, how much money has each?

10. A rectangular garden plot is 8 ft. longer than it is wide. Its perimeter is 332 ft. What are the dimensions?

11. The sum of three numbers is 392. The second is five times the first, and the third is eight times the first. Find the numbers.

12. A rope 90 ft. long is cut into two parts so that the longer part is three times as long as the shorter part. Find the length of each part.

13. Eight times a number decreased by one fourth of itself is 124. Find the number.

The median number of rights on this test is 9.

The method of determining the time on each of these tests was to stop the work as soon as two or three of the pupils had finished. In this way the test gave a measure of each pupil in a class. A stop watch was used, so that the results obtained would not vary greatly from class to class. The attempt was made to get the time factor as nearly correct as possible, but it is probable that this will have to be further adjusted if the tests are given to a larger number of pupils than the three hundred that were available in this experiment. The importance of the time element, however, should not be overemphasized.

Similar kinds of practice tests were made for the other types of equations, such as $2x = 60$, $x - 2 = 3$, and so on. As a result the tests provided the basis for a fairly correct judgment as to the approximate amount of time necessary to secure a reasonable degree of mastery of a type like $x + 3 = 6$, and they also assisted in improving instruction.

4. COMPOSITE TESTS

At the end of thirteen lessons on equations two composite tests were given on the entire topic. One test was on formal equations, and the other was on the translation of statements expressed in words into algebraic language, as described on page 26.

These composite tests included all the various types of equations or problems which had thus far been considered in the course, and which at the same time seemed to be of fundamental importance. The pupils were given enough time to finish all the problems up to the limit of the time set for the class period, which was forty-five minutes. The main idea was to discover, after thirteen lessons had been spent upon the topic, what the pupils could do with each type of exercise or problem that seemed important enough to be included. The results were then studied, and the conclusions given in the next chapter were deduced.

Class records were kept for each of the practice tests and composite tests. The results were charted on large sheets of paper with the names of pupils in a column at the left and the numbers of the problems at the top. Exercises incorrectly solved were marked by a cross, and those which were not attempted were designated by a zero. From such a record a teacher can see at a glance where pupils have succeeded and where they have failed, and can begin at once upon remedial work. The norms for future reference established from a study of the results of these tests were used to stimulate pupils to improve their records.

The class record sheet on the following page shows the actual results in "attempts" and "rights" for a class of thirty-one pupils on the first composite test in the solution of the thirty-one simple equations given on page 32.

No zeros appear in the illustration given because each pupil attempted all the exercises.

Number of Pupils	Numbers of the Equations																															Attempts	Rights
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
1												X	X		X												X					31	27
2					X	X	X		X		X	X	X	X		X	X	X	X	X	X	X	X	X		X	X	X	X		X	31	9
3				X			X		X				X		X		X	X			X		X			X	X			X	X	31	18
4								X	X		X		X		X	X	X	X	X	X		X	X	X	X		X	X		X		31	14
5																			X				X									31	29
6							X	X	X				X	X																		31	26
7						X	X		X		X	X						X	X	X			X			X				X	X	31	19
8								X	X				X						X			X	X	X		X	X					31	21
9								X					X	X													X	X		X	X	31	24
10							X		X			X						X												X		31	26
11						X		X	X												X	X				X	X	X				31	23
12											X	X	X	X	X	X		X	X				X	X	X		X	X	X	X		31	16
13																												X				31	30
14							X					X	X			X	X		X	X			X				X	X	X	X		31	19
15						X	X	X	X		X	X	X		X	X	X	X	X	X	X		X		X	X	X	X			X	31	11
16																						X								X	X	31	28
17									X		X				X	X	X		X	X	X	X	X	X	X	X	X	X		X	X	31	14
18						X	X		X		X		X				X	X	X		X					X	X		X	X		31	18
19					X	X		X	X				X		X				X	X							X	X	X	X		31	19
20						X	X		X		X		X				X	X	X	X			X	X	X	X		X	X			31	16
21									X			X															X			X	X	31	26
22					X		X	X	X	X		X	X		X				X	X	X	X	X				X	X	X	X	X	31	13
23													X					X									X					31	28
24				X									X										X									31	28
25			X					X	X				X					X									X			X	X	31	23
26						X							X														X	X		X	X	31	25
27									X				X				X	X					X				X	X	X	X		31	22
28						X		X						X	X	X	X	X	X				X	X			X	X		X		31	18
29			X						X	X	X	X	X	X	X	X	X		X	X		X	X				X	X	X	X		31	13
30						X		X	X		X	X	X	X	X							X	X		X	X		X	X		X	31	16
31							X		X				X		X		X			X					X	X				X	X	31	21

SAMPLE RECORD SHEET

CHAPTER IV

THE RESULTS OF THE TESTS IN DIAGNOSIS

1. What the Results Reveal

The main value of the tests from the standpoint of diagnosis lies in the fact that some of the results reveal the pupils' errors and often show why the errors were made. The teacher can easily utilize these findings, aided by the pupils' introspections, to improve future instruction. As we shall see, such a procedure may make some of the tasks in the tests easier, and may thus shift their position on a scale of difficulty.

It is unfortunate that it was not possible in this experimental work to consult all the pupils to whom the tests were finally given. Such a procedure would have enlisted their help in discovering the main causes of their difficulty with certain problems, but this was impossible. It is evident that the teacher who is giving the tests to a class is not thus handicapped.

Three typical tests have been selected to illustrate the method that teachers may use in bringing out characteristic diagnostic features. The method of using these results in improving instruction will be illustrated, and certain recommendations will be made.

2. Order of Difficulty of the Items in the Tests

The exercises and problems in the tests were originally arranged in what was thought to be an order of increasing difficulty. This was done partly through the information brought to light by the use of the timed practice exercises previously described and partly through conferences with pupils and teachers in which difficulties of one kind or another were discussed.

In the three tests which follow, the numbers in parentheses indicate the order of increasing difficulty as estimated at first, while the numbers in the right-hand column indicate this order as determined experimentally. A study of these two columns will reveal to what extent a teacher's subjective judgment, or more or less preconceived opinion, must be changed in the face of objective and more scientific procedure.

It is evident that the order of the exercises or problems as now given is not absolutely the order of increasing difficulty, but it is more nearly so than it would be if the single judgment of one teacher had been taken. The difficulty of each exercise or problem was determined on the basis of the per cent of pupils solving it, as described on page 48.

It should be stated in this connection that some of the exercises and problems in these tests are open to the just criticism that they tend to perpetuate the obsolete. The tests represented the current usage at the time they were made, but they should be modified in the light of later studies.

TEST ON THE EQUATION

Solve for the value of the unknown in each of the following equations:

(1) **1.** $x + 5 = 9.$
(2) **2.** $2\,z = 10.$
(3) **3.** $4\,x + 5 = 17.$
(4) **4.** $x - 3 = 4.$
(5) **5.** $2\,y - 3 = 9.$
(10) **6.** $8\,x = 5\,x + 12.$
(6) **7.** $18\,x + 3\,x - 16\,x + 4\,x = 18.$
(14) **8.** $\frac{1}{5}\,x = 6.$
(8) **9.** $3.4\,y - 1.2\,y + 4.8\,y = 70.$
(11) **10.** $6\,x + 3 = 2\,x + 35.$
(16) **11.** $\frac{3}{2}\,x = \frac{12}{2}.$
(15) **12.** $\frac{3}{4}\,y = 6.$
(7) **13.** $13\,x + 12 - 3\,x - 2 = 20.$
(17) **14.** $\frac{1}{3}\,x + \frac{1}{2}\,x = 30.$
(21) **15.** $2\,x + 6\frac{1}{2} = 12.$
(22) **16.** $5\,x - 8\frac{1}{2} = 9.5.$
(24) **17.** $8\,x + 30 - 4\,x = 60 + 3\,x.$
(12) **18.** $6\,y + 4 = 52 - 2\,y.$
(18) **19.** $\frac{1}{4}\,x - \frac{1}{8}\,x = 2.$
(19) **20.** $\frac{1}{3}\,y + 2 = 5.$
(25) **21.** $4.3\,x + 0.24 = 8.84.$
(29) **22.** $\frac{2}{9}\,x + \frac{1}{6}\,x = \frac{1}{18}\,x + \frac{1}{3}.$
(9) **23.** $9.7\,x + 7.5 + 3\,x - 3 = 29.9.$
(20) **24.** $\frac{1}{3}\,z - 5 = 9.$
(13) **25.** $12 - 8\,x = 3 - 2\,x.$
(26) **26.** $0.4\,x - 5 = 3.8.$
(23) **27.** $\frac{1}{2}\,x = \frac{7}{4} - \frac{1}{3}\,x.$
(27) **28.** $8\,y - 14 - 3\,y + 9 = 0.$
(31) **29.** $\frac{5}{6}\,x + 2\frac{5}{12} = 3 + \frac{1}{4}\,x.$
(28) **30.** $\frac{1}{4}\,x + 4\,x - \frac{5}{3}\,x = 49 - 1\frac{1}{2}\,x.$
(30) **31.** $\frac{5}{2}\,x - 1 = \frac{17}{3}\,x - \frac{5}{3} - 2\frac{1}{2}\,x.$

This is a composite test on formal equations. The numbers in parentheses show the prognosticated order of difficulty; the others show the final order.

3. THE RESULTS OF THE EQUATION TEST IN DIAGNOSIS

The first test on the equation (p. 32) was given to 1204 pupils after thirteen lessons in the subject had been given. It will be observed that the order of difficulty of the exercises was not always correctly estimated in advance; for example, we see that (7), which indicates Ex. 7 in the list as first arranged, is six places from its position in the final order, (8) is one place out, (9) is fourteen places out, and various others were also misplaced. The most important consideration, however, is the reason why certain exercises turned out to be so much harder for the pupils than was anticipated, and why certain others which were at first judged to be about equal in difficulty were placed so far apart by the pupils. Such considerations also assist in discovering the general causes of failure in work of this nature.

Why, for example, was (1) easier than (4)? The former was solved by 97.5 per cent of those attempting it, while the latter was solved by only 93.4 per cent of them. Is it because the process of subtraction, which must be used in solving (1), is generally easier than addition, which must be used in solving (4)? Or, on the other hand, is it because we teach the pupils how to solve the first case better than we do the fourth? It seems probable that (4) is intrinsically harder than (1), and that an attempt to teach it so as to make it as easily solved as (1) would be unwise. The amount of emphasis in our teaching must be governed by common-sense rules as well as by supposed psychological ones, and the two, unfortunately, are not always identical.

Again, why was (14) solved by 82.8 per cent of the 1204 pupils who tried it, while (19), a somewhat similar exercise, was solved by only 64.3 per cent? Was the added difficulty due to the fact that the extra operation of subtracting 2 was involved, was it due to the mere fact that another connection had to be made, or was it because the teaching was not adequate? Indeed, we might reasonably ask whether it should be expected that (19) should be as easy for any group of pupils to solve as (14).

In view of the fact that (17) was easier than (19), it might not be safe to draw any conclusions without having all the facts before us. Who, at present, really knows which of the two exercises should be the harder, and why? This is merely typical of some of the problems that await solution. The whole question of emphasis in our teaching needs to be better understood.

It is also interesting to note that (6), involving at least one more fundamental operation than (14) or (19), was comparatively easy for all pupils. Is the ease with which this exercise is solved due to the kind of operations involved or to an increased emphasis in teaching this type? If we could ask the pupils themselves to testify, we might be able to give a better answer.

Similarly, both (21) and (22) involve two operations, and the second term in each is a mixed number; yet they prove to be easier for the group of pupils tested than (19). Is it possible that the group which was tested does not represent general conditions, or are these questions to be answered in some other way? Some of the difficulty may be explained by noting that certain operations combine in an exercise better than others, but there is still much to be learned about special difficulties.

We have ordinarily thought that the appearance of decimals in an equation was a stumbling block to the pupil. Is this because they are intrinsically difficult or because in this country we have given too little emphasis to decimals? We see that (8) involves decimals and requires addition, subtraction, and division; yet it was solved by 80.0 per cent of all the pupils attempting it. On the other hand, (9), which might not appear much harder than (8), although placed after (8), was solved by only 53.3 per cent of the total number attempting it. It involves almost the same operations as (8), — why, therefore, is it harder? In (8) each of the terms on the left side of the equation contains the unknown, the coefficient in each term being a decimal, and the number on the right is an integer. On the left side of (9) we have two terms containing the unknown, one a decimal and the other an integer, and one of the constant terms is a decimal and the other is an integer; also, the number on the right is a decimal. Which of these differences accounts for the increase in difficulty? As we shall see, it cannot be due entirely to the decimals.

While it is impossible to give complete answers to such questions in this study, it is proper to raise them and to give the best answers we can from data available, leaving to future studies the question of a more satisfactory decision. As has been previously stated, some of the questions can be answered only with the help of the pupils to whom the tests are given; and if the best results are to follow, the teacher should discuss the results with the pupils immediately after the test has been given.

We shall now consider what the actual results show in the case of some of the exercises and problems. In the solution of equations no more difficult than the first five or ten of the revised list (p. 32) the pupils performed one operation about as well as another, provided the teaching was carefully done; but when the equations began to involve more terms and operations, or to become more complex in any other way, a large number of incorrect solutions appeared. It is remarkable how pupils can find so many incorrect answers to an equation. No attempt will be made to list all the absurd answers given by only one or two pupils, but the difficulties will be grouped and discussed in order to show what kind of remedial work is needed in such cases.

1. In the case of (7) the largest source of error appeared in purely careless mistakes with signs; the second source was in adding terms containing knowns and unknowns indiscriminately; and the third was in failing to attempt the exercise.

2. In (9), which was similar to (7) except for decimals, the chief source of error was found in adding terms containing knowns and unknowns. The common mistake was to write $9.7\,x + 7.5 = 17.2\,x$ and $3\,x - 3 = 0$, thus disregarding the fact that the terms were not similar. The second source lay in thinking that $9.7\,x + 3\,x = 10\,x$, $7.5 - 3 = 7.2$, and in writing $1.27\,x$ instead of $12.7\,x$, which displays either a lack of understanding of decimals or else carelessness. As in (7), a large group failed to attempt the exercise.

3. In (12) nearly all the trouble was caused by the pupils' writing $4\,y = 48$, whence $y = 12$, no attention being paid to the minus sign on the right. This may indicate merely a lack of proper teaching of this type, but it would not be safe to make such an assertion. It may, for example, mean that the initial teaching was satisfactory but that the drill was insufficient. The only other feature worth mentioning here was, as has been remarked in the previous cases, that many pupils gave up entirely, — a thing that happened altogether too often. It may indicate that the pupils became

discouraged because they were not properly taught. Altogether there were twenty-four different ways in which the pupils tried to solve this exercise.

4. There were twenty-five different incorrect methods used in trying to solve (13). Here again a large part of the errors were due to a wrong use of signs; but the main source of failure was a lack of knowledge of how to finish the solution, or perhaps a lack of desire to use the knowledge possessed. In this exercise in particular a number of pupils gave such results as $x = \pm\frac{9}{10}$, $x = \pm\frac{10}{9}$, and $x = \pm\frac{2}{3}$, showing carelessness or ignorance not only with reference to signs but also with reference to the law of division. A rather large number stated that $6x - 12 = 3$, $6x = 15$, and $x = 2.5$, showing again a careless use of signs. This probably indicates insufficient preparation rather than lack of desire to succeed.

5. In (19) the most common source of error was the inclination to write $y = 5 - 2 = 3$, and the second was to take the equation $\frac{1}{3}y = 3$ as leading to $y = 1$. The third cause of failure was that the pupils simply abandoned the work at some point in the solution, evidently because they were not sure what they should do next. This seemed to be due entirely to a lack of preparation.

A great deal of improvement will be made in the tests if we can group the work satisfactorily around certain central principles. For example, (1), on page 32, involves subtraction; (2), division; (3), subtraction and division; (4), addition; (5), addition and division; and so on. If we keep the exercises in this order, it will be easier to discover precisely what operations cause difficulty, and then we can give more emphasis to them in our teaching.

Again, the solution of (6) makes necessary the collection of similar terms in the left member. These terms all involve the unknown, and the coefficients of all the terms are integers. Similarly, (8) involves the unknown only on the left side of the equation, but the coefficients of all the terms are decimals. The solution of (7) makes necessary the collection of terms containing both knowns and unknowns, where both the coefficients of the terms in the unknown and the known terms themselves are integral. We might, therefore, group (7) with (6) and (8). In (9) there is a mixture of terms containing knowns and unknowns, one of the known terms and the coefficient of one of the terms containing the unknown being integers, while the other known term and the coefficient of the other term containing the unknown are decimals. This equation, therefore, might also be included in the above group with (6), (7), and (8).

In (10) and (11) the unknown appears on the right side of the equation, and the operation of subtraction is involved; while in (12) and (13) the unknown appears on the right side of the equation, and the operation of addition is involved. If, as previously indicated, the main source of error is a careless regard for signs, then we must be careful to teach pupils to be more accurate in solving equations similar to those involved in (10), (11), (12), and (13). If we follow our grouping plan here, some help may be given. In this way we might arrange all the exercises and problems in groups as far as possible for diagnostic purposes.

These results show that, for some pupils, the solution of relatively easy equations is more difficult than is often recognized by high-school teachers. It is probable that some of the teachers, whose pupils were tested in this study, rely too much upon transfer.

It is also interesting to observe that when certain exercises in the various tests were given as part of a composite test at the end of the year's work, the degree of difficulty of the tasks, which was determined from the percentage of correct solutions, as explained later, was found to vary slightly. Such a variation was to be expected, since the tests covering the year's work were not given to as large a group of pupils as the one to which the above test was given, and, as already suggested on page 31, the position of tasks on a scale of difficulty may be changed by utilizing the results of previous tests to improve instruction. This will explain, however, certain variations when we come to discuss the difficulty of the tasks in connection with the scales that are given in Chapter VI.

We need to analyze more fully than we generally do the various algebraic topics and to select from them the unit skills or difficulties to which we must pay particular attention in teaching. We can then try these out with the pupils. In this way we shall be able to classify difficulties so that the lesser ones will soon be overcome and the necessary drill and remedial instruction can be given for those in which the degree of difficulty is greater.

4. Test on Applied Problems

The following composite test on applied problems was given to 1196 pupils immediately after the above test on formal equations was finished. Here also it is found that the use of the more scientific method in arranging the problems in the order of increasing difficulty is the best one.

TEST ON APPLIED PROBLEMS

Write the answer or the equation that expresses the conditions of each problem, but do not solve any equation:

(2) 1. A boy had a marbles and lost c marbles. How many had he then? *Equation*_________

(1) 2. What is the cost of 9 pencils at k cents each? *Answer*__________

(3) 3. A man is x years old. How old will he be y years from now? *Answer*__________

(11) 4. The sum of three numbers is 240. The second is four times the first, and the third is seven times the first. Find the numbers. *Equation*_________

(7) 5. A room is l feet long and w feet wide. How many feet of border should such a room require? *Answer*__________

(8) 6. A brick house is worth three times as much as a frame house. If both together are worth $4080, how much is each worth? *Equation*_________

(10) 7. A pole 30 ft. long is broken into two parts so that the longer part is twice the shorter part. Find the length of each part. *Equation*_________

(6) 8. If four times a number increased by 6 is 42, what is the number? *Equation*_________

(15) 9. The number of representatives and senators together in a certain legislature is 142. The number of representatives is 26 more than the number of senators. Find the number of each. *Equation*_________

(4) 10. Find two consecutive numbers whose sum is 45. *Equation*_________

(13) **11.** Find the sides of a triangle in which the second side is 4 ft. longer than the first, the third side 6 ft. longer than the first, and the perimeter 40 ft. *Equation*_________

(5) **12.** Find two consecutive even numbers whose sum is 94. *Equation*_________

(12) **13.** Nine times a certain number increased by one third of itself equals 56. Find the number. *Equation*_________

(14) **14.** A quadrilateral *ABCD* has a perimeter of 42 in. The side *BC* is twice as long as *AB*, the side *AD* is three times as long as *AB*, and the side *DC* is 7 in. longer than *AB*. Find the length of each side. *Equation*_________

(9) **15.** A rectangular flower bed is 4 ft. longer than it is wide. Its perimeter is 40 ft. What are its dimensions? *Equation*_________

(16) **16.** A man divided his farm of 13,000 acres among his three boys. The first received twice as much as the second and 2000 acres more than the third. How much did each boy receive? *Equation*_________

(17) **17.** A boy had 7 times as many pennies as his sister. After he had given 9 pennies to his sister, he still had 36 pennies more than she. How many pennies had each at first? *Equation*_________

The numbers in parentheses show the prognosticated order of difficulty; the others show the final order.

5. THE RESULTS OF THE PROBLEM TEST IN DIAGNOSIS

There were not so many changes in the final order of the problems in the above test, but two or three outstanding exceptions will be considered. Thus, (4) and (5), relating to two consecutive numbers, were more difficult than was expected. This was probably due to the fact that the pupils did not understand the word "consecutive." In (4) their solution was often as follows: "Let x represent the first number, then $2x$ will represent the consecutive number, and hence $x + 2x = 45$." This error was the most common one found in all the cases in which the problems were incorrectly solved. Similarly, in (5) they followed (4), or else solved as follows: "Let $2x$ represent the first even number; then $4x$ will represent the consecutive number, and hence $2x + 4x = 94$."

The next to the largest source of error in (4) was in writing $22 + 23 = 45$, which indicates a knowledge of the word "consecutive" but shows a lack of knowledge of the algebraic method. There is some question, however, as to whether the above solution is incorrect. It may be that the pupils who solved the problem in this way need further training in the use of algebra, but we must avoid the charge that "algebra is arithmetic done in a harder way." The third source of error was in writing $a + a = 45$ and $x + 1 = 45$. Besides these three large sources of error, twenty-five others, which included various freak methods of solution, were found. In fact, if one examines a sufficient number of results, it will be found that the pupils have made every conceivable mistake, which, after all, is only what is to be expected in a large group. In the same way, the next to the largest source of error in (5) was found in such forms as $46 + 48 = 94$ and $c + c = 94$. There were altogether twenty-seven other incorrect solutions.

If problems about consecutive numbers are worth teaching at all, we must learn to teach them better. The main source of error in the two problems can doubtless be largely eliminated by an inductive approach. If the teacher gives any number, say 2, and asks the pupils to state the consecutive number, they will nearly always give 3 as the answer; and if asked for the consecutive even number, they will give 4. A number of such cases ought to furnish a basis for bridging the gap between the arithmetical method and the algebraic method of expressing such an idea.

In this test (9) proved to be fifteenth in the final order. The reason for this is easy to explain. Most of the pupils failed because they used half the perimeter of the flower bed instead of the whole perimeter. In cases in which they used x to represent the width and $x + 4$ the length, the most frequent error was to write $x + x + 4 = 40$. The frequency of such errors can doubtless be greatly reduced if in this type of problem the teacher will insist on the pupils' drawing a rectangle and labeling each dimension before writing the equation. Another error that was made by a large number of pupils was in writing $x + 4x = 40$, and other frequent errors were as follows: $x + 4 = 40$, $4x + 4 = 40$, $4x = 40$, $2x - 8 = 40$, and $2x + 8x = 40$. It is probable that the frequency of all these errors can be materially reduced by a greater insistence upon the drawing and labeling of a figure; at least this has been the experience in my own classes. If this is generally true, the order of difficulty of many of the exercises and problems may be changed, but this need not give us any concern. We shall simply have to come to some agreement as to where the emphasis in teaching must be placed.

In spite of the difficulty experienced by the pupils in solving (9), a similar one, (7), proved to be very easy. It is hard to explain why this was so. One would think that the pupils who wrote $2x + 4 = 40$ in (9) would write $l + w$ in (7), but they did not. Perhaps the fact that there was a 2 before the x in $2x + 4 = 40$ led the pupils to think that they had doubled all the dimensions.

The order of (14), (16), and (17) remained unchanged in the final list, showing that they were properly placed. There is certainly no doubt that (17) is the hardest in the list. It was solved by only 50 per cent of a large group of mathematics teachers at a certain meeting, which indicates that we should confine our list of applied problems to a more reasonable selection. We should be careful to teach only the ones that appear to have the best chance of being of real value to the pupil. The theoretical side of algebra may as well be presented by means of interesting, real, and practical problems as by any other type.

Since these tests appeared, various teachers have criticized (8) and (17) as being poorly chosen, for the reason that they are not genuine. Here, as in other instances throughout the tests, it seems certain that some of the work may well be replaced by better material. This, however, does not detract from the value of the method of testing herein recommended.

6. TEST ON PROPERTIES OF ANGLES

The following composite test, which consists largely of definitions and constructions, was given to 992 pupils. All the definitions except one proved to be relatively easy, (1) appearing as 11 of the final list. This means that the definition of an angle was the hardest in the entire test, which is due to the fact that the conventional definition of an angle is longer and more involved than those of the other terms. It looks as if the mere fact that "angle" has been traditionally defined before "right angle" was the reason for assigning to it the first place in the original list. Such a result lends added weight to the practice of not trying to define "angle" at all, or at least of defining it only informally.

In the remainder of the test certain tasks are fundamental to others. For example, in the original test (12) was the basis for (19), (14) was the basis for (16) and (18), and (15) was the basis for (17).

TEST ON PROPERTIES OF ANGLES

Complete the following statements:

(2) **1.** A right angle is an angle..............................

(3) **2.** An acute angle is an angle..............................

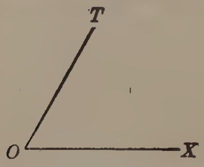
(8) **3.** Construct an angle of 56° on the working line AB by using only the protractor.

Complete the following statement:

(4) **4.** A straight angle is an angle..............................

(14) **5.** Bisect the line segment AB.

(15) **6.** Bisect the angle XOT.

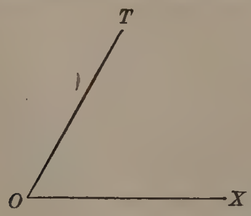


Complete the following statements:

(5) **7.** An obtuse angle is an angle..............................

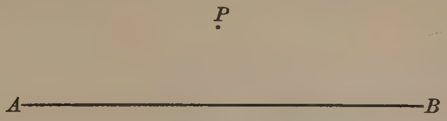
(6) **8.** A perigon is..............................

(7) **9.** Measure XOT with the protractor and express its size in degrees.



Angle $XOT =$ degrees.

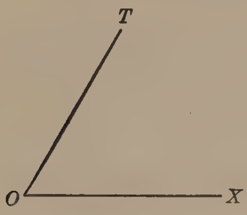
(12) **10.** Construct a perpendicular to a given line AB from an external point P.

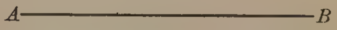


Complete the following statement:

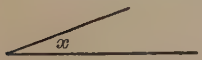
(1) **11.** An angle is...

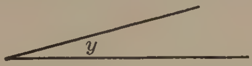
(9) **12.** Construct an angle equal to the given angle XOT by using only the ruler and compasses.



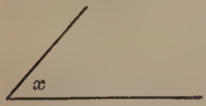


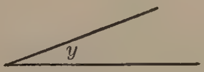
(10) **13.** Show the sum of angles x and y on the line AB.





(11) **14.** Show the difference between angles x and y on the working line AB.



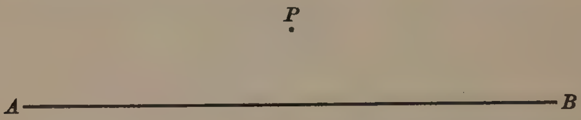


(16) **15.** Draw an acute-angled triangle ABC and construct the perpendicular bisectors of the three sides. Leave all the construction marks on the paper.

(13) **16.** Construct a perpendicular to the line *AB* at the point *B*.

(17) **17.** Draw an acute-angled triangle *ABC* and construct the bisectors of the three angles of the triangle.

(20) **18.** Construct a line through the point *P* parallel to the given line *AB*.



(18) **19.** Draw an acute-angled triangle *ABC* and construct the three medians of the triangle.

(19) **20.** Draw an acute-angled triangle *ABC* and construct the three altitudes of the triangle.

(22) **21.** Construct a square whose side is 3 cm. long.

(21) **22.** Two parallel lines are cut by a transversal so as to form two corresponding angles of $(3x - 150)°$ and $(x + 80)°$. Find the value of x and the size of each angle. *Answer* __________

The numbers in parentheses show the prognosticated order of difficulty; the others show the final order.

7. THE RESULTS OF THE TEST ON THE PROPERTIES OF ANGLES IN DIAGNOSIS

The final arrangement in the order of increasing difficulty in the above test shows that in the original test the more difficult tasks had been properly left toward the end of the list. This is clearly an indication that basic constructions may be performed by pupils who are unable to apply them in a more complicated situation until they have been definitely trained to do so. Thus, 768 pupils out of 992 constructed correctly the perpendicular to a given line from an external point *P*, while only 400 out of the 992 were able to utilize this construction so as to construct correctly the three altitudes of a triangle. In fact, the majority of those who failed to construct the three altitudes could not even draw one correctly. Many pupils succeeded in constructing one correctly, — the one from the vertex of the triangle directly above the base, which was in the same relative position as the point *P* in the basic construction. This raises a question as to our method of teaching the pupils in the first place. One reason why the pupils could construct only the one altitude where the vertex is directly above the base was that they had never been taught to construct a perpendicular from a point *P* to a given line in any other position than that shown in (12). This is due largely to the fact that textbooks in general give only one position for the construction, because it makes a more conventional figure. If it is important that this construction be applied as in (19), it should be performed with the point and the line in several different positions.

In (16), where the pupil was expected to apply what he had learned in (14), the chief source of error was an inability to complete the construction successfully either because the pupil did not know what "perpendicular bisector" meant or, knowing that, because he did not see how to apply the result of (14) to the construction in (16). Some drew only one bisector, a few more drew only two, and some, of course, drew all three. The other sources of error, outside of those mentioned above, were as follows: a great many confused (16) with (17), and constructed the bisectors of the angles; approximately the same number constructed the medians; somewhat fewer constructed the altitudes; and in some cases pupils who could probably have performed the constructions correctly became confused over the concepts "bisector of an angle," "median," "perpendicular bisector," and "altitude," and thus interchanged the constructions.

In (18), where the pupils were expected to apply (14) and (16), we find that the chief source of error was the inability to apply these constructions, either because they did not fully understand what a median was or, knowing that, because they could not apply their knowledge. It is evident that the pupils had not been thoroughly prepared for this work. In most cases they made no attempts at all, while in others they had no intelligent system that could be discovered. In a great many cases the pupils constructed altitudes, bisectors of angles, or perpendicular bisectors, but not medians. Here again the pupils often succeeded in constructing the median to the horizontal base but did not appear to know how to construct the others. This is clearly an indication that in teaching construction problems we should see that pupils perform the constructions with the lines in positions other than the conventional horizontal one usually found in textbooks.

In (17), where it was intended that the pupils should apply what they had learned in (15), it was found that the chief source of failure was the absolute inability to do more than draw the triangle and put down a few unintelligible marks. In fact, more pupils failed to solve (16), (17), (18), and (19) because they could not apply what they had learned than for any other reason. It seems probable that the trouble lies mainly in the teaching, because in the case of these problems nearly all the failures outside of pure ignorance of the method were due to confusing bisectors of the angles with altitudes, medians, and perpendicular bisectors of the sides; that is, these concepts were not clearly developed in the case of the pupils who furnished the data for this study. If these so-called concurrent-line problems are worth teaching, we must be careful to teach the basic concepts better than was done in the case of many pupils in this study.

8. CONCLUSIONS

The results from the tests discussed above lead to the suggestion that we should give more attention to what we consider fundamental in a course, and that we should seek a greater mastery of these fundamentals, even if we have to shorten the range of the subject matter that the course now covers. This does not mean that we should seek a complete mastery on the part of all who study the subject, which is both impossible and unnecessary, but that we should be able to secure greater mastery than is shown by the results of most tests. The desirable mastery of the abilities in mathematics will be better secured by a wider and more intelligent use of diagnostic tests.

CHAPTER V

THE CONSTRUCTION OF THE SCALES

1. The General-Mathematics Scales

In connection with this study the exercises and problems in the various composite tests, such as those given on pages 32, 38, and 42, were used as a basis for the construction of a series of seven general-mathematics scales, which are designed to be given at intervals during the year. Two different forms of each of these scales were prepared, although only one form of each scale is here given for purposes of illustration (see pages 58–84).

The same tests were used also as a basis for the construction of three different forms of composite scales (see pages 85–100) covering the work of the entire year, and intended to be given as final tests at the end of the year's work. A key for scoring, with a list of acceptable answers, was also prepared to accompany the scales.

In the first place, it is assumed that the scales should measure some ability which the pupils who are studying a course in general mathematics ought to have. If a teacher does not consider this ability fundamental, these scales are not the ones for him to use. In the second place, no tasks were included in the scales which were too difficult for all pupils; that is, so difficult that a pupil was seldom found who could do them in the allotted time.[1] Finally, no tasks were included which were too easy; that is, so easy that they were almost never missed by a pupil.[2]

2. How the Scales were Constructed

Each scale is composed of thirty exercises or problems, arranged in the order of increasing difficulty. The advantage of this order is, of course, that it enables a teacher to determine how far along the scale an individual pupil can go.

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[1] For a list of problems and exercises of this type see page 113.

[2] For a list of such cases see page 111.

The total number of correct answers or solutions for each exercise or problem in the composite tests was counted, and from these was computed, as shown in the table on page 104, the percentage of pupils who completed each one correctly. This gave a score indicating the percentage of possible correct responses to each exercise or problem, the score serving also as a measure of the difficulty of each task. We may therefore define the ability of the pupil in terms of the difficulty of the tasks in the scale which he is able correctly to complete. We can then define his ability in terms of its relative position with reference to the abilities of the pupils who are in his group.

The preceding statement is based upon the assumption that the ability of pupils to solve the exercises or problems is distributed within the ninth-grade group according to the normal curve of distribution. According to this assumption we ought to find that most of the pupils solve approximately the same number of exercises or problems, this number corresponding closely to the median of the group. Furthermore, we ought to find a very small number at one end who can solve only a few, and another small group at the other end of the distribution who succeed in solving a very large number. That is to say, we ought to find a rather symmetrical distribution such as is given by the normal curve of frequency.

The difficulty of a task may then be represented by means of a point on the base line of the normal curve of distribution such that if a perpendicular is erected at that point it will cut off a part of the area between the base line and the curve whose ratio to the total area under the curve will be equal to the percentage of pupils who are able to complete correctly tasks of that degree of difficulty.

For example, an exercise which is solved correctly by 50 per cent of the pupils trying it falls at the midpoint of the base line where the perpendicular divides the normal curve of distribution into two equal parts. This represents the median achievement of the group. An exercise which is solved correctly by only 25 per cent of the pupils attempting it falls at one quartile deviation unit below the median point. A perpendicular erected at this point will cut off on the right, between it and the median perpendicular, 25 per cent of all the cases. In like manner an exercise which is solved correctly in 75 per cent of the attempts falls at one quartile deviation unit above the median point. A perpendicular erected at this point will cut off 25 per cent of the whole number of cases between it and the median perpendicular. Between these two perpendiculars erected at the quartiles will be included 50 per cent of all the cases.

3. THE SCALE UNIT

The unit selected in constructing the scales is the median deviation, or probable error (P. E.). It is the unit which has generally been used in the construction of scales, and is a measure of variability. This unit is defined as the median amount of deviation from the central tendency, which in this study is referred to as the median.

In terms of P. E. distances an exercise that is solved by only 25 per cent of the pupils attempting it is one unit harder than the one at the median point, and hence should be located at +1 P. E. from it. Similarly, a problem that is solved by 75 per cent of the pupils attempting it is one unit easier and is therefore located at −1 P. E. from the median point. We know also that if a perpendicular is erected at a distance of 2 P. E. on either side of the median point on the base line of a normal curve of frequency, it will include with the median perpendicular 41.13 per cent of all the cases; if erected at a distance of 3 P. E., it will include 47.85 per cent of them; and if erected at a distance of 4 P. E., it will include 49.65 per cent.

Although theoretically the normal distribution curve and the base line never meet, no matter how far they are extended, we have assumed here that they meet at a distance of 4.6 P. E. from the median point, since the perpendicular erected at that point on either side of the median cuts off only 0.1 per cent of all the cases. For practical purposes we may therefore neglect this part of the distribution, particularly as the groups at either end have been somewhat diminished by the fact that, as has already been stated, the extremely easy and the extremely difficult exercises or problems were excluded from the scales.

4. DIFFICULTY OF THE ITEMS OF THE SCALES

The difficulty of each exercise or problem in terms of P. E. is easily determined by utilizing the table perfected by Professor Buckingham.[1] Figures indicating the scores in percentage and the corresponding P. E. values for all the exercises and problems in the tests are given in the table on page 104.

In this way the percentage deviation was determined from the percentage of pupils who solved each exercise or problem. These values were then converted into P. E. distances from the median achievement of the entire number taking the test, and P. E. values for each of the items were determined.

5. SCALE SCORES

From these P. E. values there were determined, by a method perfected by Professor Van Wagenen and others, scale scores or values for each exercise in the tests. The seven scales were made from the tests listed on page 102, as follows:

1. Series A, from tests 1 to 3 inclusive.
2. Series B, from tests 4 to 6 inclusive.
3. Series C, from tests 7 to 9 inclusive.
4. Series D, from tests 10 to 12 inclusive.
5. Series E, from tests 13 to 15 inclusive.
6. Series F, from tests 16 to 19 inclusive.
7. Series G, from tests 20, 21, and 24.

Tests 22 and 23, on topics 17 and 18 (see page 21), were not used in constructing the scales because they were not given to enough pupils to justify their inclusion. These will be utilized later, when they have been given to a sufficiently large number of pupils.

6. HOW THE SCALE SCORES WERE OBTAINED

The method of obtaining the scale scores is based upon the assumption that when the items are scaled, the total number correctly solved is numerically equal to the number of the item corresponding to a type which the pupil can do with 50 per cent of accuracy. If, for example, the pupil solves correctly a total of 20 items from Series A, Form I (page 58), he will probably be able to do the twentieth exercise with 50 per cent of accuracy. A scale score of 80 indicates that a pupil can do items of difficulty 80 with 50 per cent of accuracy. It is also an indication that he will probably do items of difficulty 70 with 75 per cent of accuracy, and those of difficulty 90 with 25 per cent of accuracy.

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[1] B. R. Buckingham, "Spelling Ability: its Measurement and Distribution," *Contributions to Education*, No. 59. Teachers College, Columbia University.

7. THE ZERO POINT OF THE SCALES

Since the exact determination of a zero point is not possible with the data at hand, and since for our purpose its exact location is relatively unimportant, we arbitrarily define the zero point of each scale as 80 below the median ability of the group.

If, then, the zero point of each scale is located at a point representing an ability 80 below the median ability of the group, the standard median is 80, or 80 deciquartiles above the zero point. This means that the scores obtained on one of the scales are directly comparable to those of another, since their zero points are the same.

8. THE USE OF THE TABLES

In order to make clear the method of using the tables in determining the difficulty of an exercise or problem by locating it on the base line of our group distribution, let us take the first exercise in Series A, Form I, given on page 58, which was (4) in the test on page 32. By referring to the table on page 102 we see that 1125 of the 1204 pupils who attempted this exercise solved it. From the table on page 104 we observe also that this is 93.4 per cent of the total number who attempted it. This means that such achievement is 43.4 per cent more than the median achievement of the entire group. The corresponding value of 43.4 per cent as obtained from Professor Buckingham's table is 2.234. But since this number is the measure of the difficulty of the exercise when compared with the median achievement of the group, it is clear that this particular exercise should be located at a point which lies at a distance of -2.234 P. E. below the median point. The scale score or value of this exercise turns out to be 58, which is given in parentheses after the number of the exercise.

9. MEANING OF A SCALE SCORE

Having assumed that the ability measured by the scales is distributed according to the normal curve of frequency, we define the score of a pupil as *that numerical value which defines a point representing his ability on the base line of the normal curve of frequency which represents the distribution of the abilities of the group.* We may then say that a pupil's score represents also the percentage of correctness with which he may be expected to perform tasks at the various levels of difficulty.

Thus, in Series A, Form I, page 58, where the items of the scale are arranged in order of increasing difficulty, we do not expect to find a pupil who can solve all the exercises or problems up to any particular point unable to do any beyond that point. He may actually solve some of the exercises in any group. For example, he may fail to solve the tenth, eighteenth, nineteenth, and twentieth, and all the rest except the twenty-second and twenty-third. Such a pupil may be expected to solve with a high degree of accuracy any exercises or problems of a degree of difficulty like that of the first ten. He may do reasonably well with items of a difficulty like that found in the second group of ten, but not so well as he did in the first ten. He will not be able, however, to solve many exercises or problems of a difficulty like that found in the third group of ten.

In certain cases a pupil's score on the above scale may indicate the percentage of accuracy with which he may solve exercises or problems of any degree of difficulty within the limits of the scale.

10. METHOD OF SCORING

There is general agreement concerning the method of scale construction used in this study, but there are few people in the field of tests and measurements who are willing to accept the method of scoring. One of the greatest objections is that it is too complicated for teachers to use. Professor Powers[1] maintains that in the case of his chemistry scales "the advantages, when the sole objective of testing is that of ranking pupils, appear to be lacking, for the correlation between point scores (number of items correct) and scale scores is nearly one."

---

[1] S. R. Powers, "A Diagnostic Study of the Subject Matter of High School Chemistry," *Contributions to Education*, No. 149 (1924), pp. 71–72. Teachers College, Columbia University.

Since the scales have been constructed by giving the scale items to a large number of pupils and determining their difficulty thereby, it is probably also true that an individual pupil may not receive fair treatment if he is judged solely by a scale score. This seems to be all the more important when we remember that pupils will actually solve some of the exercises in each part of the scale. Because of these facts, and because it would seem that the method of scoring should be one that any classroom teacher can employ, it is recommended that in each scale the score be determined by counting the number of exercises and problems correctly solved. When the method of scoring described above can be made simpler, and can be shown to be more reliable than the method of counting "rights," then its use may be recommended.

CHAPTER VI

ADMINISTRATION AND USE OF THE TESTS AND SCALES

1. Timed Practice Tests

The contents of this chapter will be of interest to teachers who follow the course of study and the methods of teaching general mathematics indicated in the preceding pages. If they follow the scheme fully, it will involve, first, the construction and use of carefully prepared timed practice tests like those given on pages 27 and 28.

In so far as these tests meet the needs of a particular class, a teacher can use them to encourage speed and accuracy (principally the latter, although both are desirable) during the process of teaching. The order of procedure would be to teach the first fundamental operation used in solving equations like $x - 3 = 6$, and then give the first timed practice test. If the results are not satisfactory, the teaching may have to be repeated for certain pupils, and the test given again. The next step would be to teach the second fundamental operation involved in the solution of equations like $2x = 6$, then give the second timed practice test, and so on. Thus, the timed practice tests can be used to guide the teacher in deciding on the amount of drill to be given on various topics. Too much drill is often given on a topic if the teacher is not careful to check the teaching repeatedly. Overlearning is to be preferred to underlearning, but there is no good reason why we should have a superabundance of the former, and we shall not have it if we check our work often enough.

Too much importance should not be attached to either the time or the median in the timed practice tests. Speed is important, but accuracy is the main thing to be secured. Median scores, however, may be used to encourage competition among pupils in the class or to stimulate a pupil to improve upon a previous score. It would seem to be much better to follow the suggestions given by

Professor Symonds,[1] who concludes that the best plan is to place at the top of the test such directions as the following: "See how many of these examples you can do in five minutes." The time should not be shortened because the brightest pupils get through quickly, nor should these pupils be forced to sit a long while and do nothing while the dullest pupils are finishing the test. This is a matter to be decided by the common sense of the teacher, who must see to it that each pupil achieves the maximum of which he is capable, whether he is bright or slow.

2. THE COMPOSITE TESTS

In the second place, teachers will wish to check their teaching further at the end of each topic, which may be done by means of composite tests. The outline of lessons given below, which was worked out by the writer for his own course, may be used as a guide in giving the composite tests. Here again the outline is only suggestive, but it is given for the purpose of encouraging teachers to plan their teaching and testing according to some definite schedule, so that real progress may be assured. Often the failure of a teacher to cover a course of study is due mainly to the lack of a definite time schedule for covering the work. In the following outline the topic numbers refer to the list on page 21:

TOPIC NUMBER	NUMBER OF LESSONS	TEST NUMBER	TOPIC NUMBER	NUMBER OF LESSONS	TEST NUMBER
1	14	1 and 2	11	10	14 and 15
2	6	3	12	8	16
3	9	4	13	9	17
4	9	5	14	7	18 and 19
5	5	6	15	10	20
6	9	7	16	10	21
7	8	8	17	8	22
8	10	9	18	6	23
9	15	10, 11, and 12	19	10	24
10	15	13			

---

[1] P. M. Symonds, "Special Disability in Algebra," *Contributions to Education*, No. 132 (1923), p. 78. Teachers College, Columbia University.

In the case of the composite tests, as with the timed practice tests, the matter of time is not to be overemphasized. It is even more important here that plenty of time be given for all the pupils to finish the tests. At least the entire recitation period should be made available for those who need it, and those who do not need it should be supplied with something else to do. It is the purpose of the composite tests to discover which exercises and problems the pupils can do and why they fail to do certain others. If we expect to realize this purpose, the pupils must be given a chance to try each exercise or problem in every test.

Experience with the composite tests[1] shows that the brightest pupils will finish many of the tests in fifteen or twenty minutes, while others will require all of a forty-five-minute period or more. Since we wish a complete measure of what each pupil is able to do, we are justified in giving him plenty of time. If the teacher desires to give relative standings based upon performance, he can have each pupil record the time required, which may be taken into consideration in making up the marks for the tests.

The following table gives the median number of "rights" on the various tests as they were worked out in this experiment:

TEST NUMBER	MEDIAN NUMBER OF RIGHTS	TEST NUMBER	MEDIAN NUMBER OF RIGHTS
1	20	12	16
2	11	13	9
3	23	14	6
4	14	15	12
5	15	16	12
6	15	17	6
7	6	18	6
8	7	19	3
9	11	20	6
10	16	21	5
11	7	24	11

These medians are probably not as high as they might be made with better teaching, but they will serve as a rough guide in estimating about how much one ought to expect of a normal class in such work. The main use of the composite tests should be diagnostic, and it is hoped that teachers will agree to consider them as such.

---

[1] W. D. Reeve, Chapter Tests on General Mathematics. Ginn and Company, 1922.

Thus, from the composite tests a teacher may determine roughly the approximate difficulty a given pupil is able to surmount, and in the case of errors the teacher will have an opportunity to discover the reasons for them and to introduce proper remedial work to overcome them. Norms for these tests are easily determined and can be used as guides for the work of future classes.

3. THE SCALES

The scales (Series A–G, inclusive) are to be given either in place of the composite tests or as supplementary tests. If the teacher is limited in time, it might be advisable to use only the scales, but on the whole it would seem advisable to use both.

Two forms of the scales were made, in order that the teacher may avoid the possibility of having one class find out what test another class has taken and thus make special preparation for it. The scales are approximately of equal difficulty, so that the teacher can give Form I to one class and Form II to another. Form II may also be used to test pupils who were absent when Form I was given. In the following pages, however, Form I alone is given as an illustration of these scales.

Finally, to provide the teacher with a method of checking the work of the entire year, the composite scales given on pages 85–100 were prepared. There are three forms of these scales, Form I, Form II, and Form III. They are approximately equal in difficulty and can be used by the teacher in at least three different classes without making it possible for the pupils to prepare directly for one scale after seeing the others. Certain fundamental problems have been used in each form, but on the whole the scales are different.

Norms for these composite scales are easily determined and can be used as guides in future work. As in the case of the other scales, the teacher should try to improve instruction until it produces a high type of result; that is, one which is commensurate with the ability of the group.

As has already been stated, in the scales the number in parentheses after the number of the exercise indicates the scale value.

GENERAL-MATHEMATICS SCALE

SERIES A, FORM I

The Equation; Linear Measurement; the Equation applied to Length

1. (58) Solve for x:

$x - 3 = 4$. *Answer.* $x =$ -------

2. (59) What is the cost of 9 pencils at k cents each?

Answer -----------

3. (60) A man is x years old; how old will he be y years from now?

Answer -----------

4. (61) Perform the following indicated operations:

$$15\,y - y + 4\,y =$$

5. (62) Solve for x:

$8\,x = 5\,x + 12$. *Answer.* $x =$ -------

6. (63) Draw a figure whose perimeter is represented by the equation $p = 6\,x$.

7. (64) The death rate in a certain city last year was 18 to each 1000 of the population. Express this ratio as a common fraction.

Answer -----------

8. (65) Add the following terms as indicated:

$$7.25\,a + 1.21\,a + 3\,a =$$

9. (66) Solve for x: $\frac{x}{5} = 6$. *Answer.* $x =$ -------

10. (67) Write but do not solve the equation which expresses the conditions in the following problem:

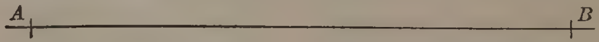
The sum of three numbers is 240. The second is four times the first, and the third is seven times the first. Find the numbers.

Equation ----------

11. (68) Solve for y:

$3.4\,y - 1.2\,y + 4.8\,y = 70$. *Answer.* $y =$ -------

12. (69) Measure the line given below and express the result in inches and also in centimeters:



$$\textit{Answers}\begin{cases} AB = \text{----------}\ \text{in.} \\ AB = \text{----------}\ \text{cm.} \end{cases}$$

13. (70) Write but do not solve the equation which expresses the conditions in the following problem:

A pole 30 ft. long is broken into two parts so that the longer part is twice the shorter part. Find the length of each part.

Equation ____________

14. (71) Solve for x:

$13x + 12 - 3x - 2 = 20$. *Answer.* $x =$ _______

15. (72) Find the value of the expression $5x - 6y$ when $x = 3$ and $y = 2$. *Answer* ____________

16. (73) Perform the following indicated operations:

$$10.5a + 4.5a - 2.3a =$$

17. (74) Solve for x:

$8x + 30 - 4x = 60 + 3x$. *Answer.* $x =$ _______

18. (75) Solve for y: $\frac{y}{3} + 2 = 5$. *Answer.* $y =$ _______

19. (76) Write but do not solve the equation which expresses the conditions in the following problem:

Find the sides of a triangle when the second side is 4 ft. longer than the first, the third side is 6 ft. longer than the first, and the perimeter is 40 ft. *Equation* ___________

20. (77) Add the following terms as indicated:

$$\frac{y}{2} + \frac{y}{3} + \frac{5y}{6} =$$

21. (78) Solve for x: $\frac{2x}{9} + \frac{x}{6} = \frac{x}{18} + \frac{1}{3}$. *Answer.* $x =$ _______

22. (79) Solve for x:

$9.7x + 7.5 + 3x - 3 = 29.9$. *Answer.* $x =$ _______

23. (80) Write but do not solve the equation which expresses the conditions in the following problem:

Nine times a number increased by one third of itself equals 56. Find the number. *Equation* ___________

24. (81) A mixture consists of alcohol and water in the ratio 4 to 8. What part of the mixture is alcohol? *Answer* ____________

25. (82) Solve for x: $\frac{x}{2} = \frac{7}{4} - \frac{x}{3}$. *Answer.* $x =$ _______

26. (83) Solve for y:

$8y - 14 - 3y + 9 = 0.$ *Answer.* $y =$ -------

27. (84) Write but do not solve the equation which expresses the conditions of the following problem:

A rectangular flower bed is 4 ft. longer than it is wide. Its perimeter is 40 ft. What are its dimensions? *Equation* ----------

28. (85) Solve for x:

$\frac{5x}{6} + 2\frac{5}{12} = 3 - \frac{x}{4}.$ *Answer.* $x =$ -------

29. (86) Solve for x:

$\frac{x}{4} + 4x - \frac{5x}{3} = 49 - 1\frac{1}{2}x.$ *Answer.* $x =$ -----

30. (87) Perform the following indicated operations:

$$4x - 6y + 5x - 3y =$$

GENERAL-MATHEMATICS SCALE

SERIES B, FORM I

Properties of Angles; Areas and Volumes

1. (59) Give the meaning of the following expression:

a^3 means ----------------------------

2. (60) Complete the following statement:

A right angle is an angle -------------------------

3. (61) Complete the following statement:

An acute angle is an angle -------------------------

4. (62) Draw an angle of 56° on the working line AB by using only the protractor.

5. (63) Complete the following statement:

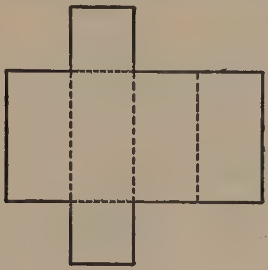
A straight angle is an angle--------------------------

6. (64) Express by an equation the area of a rectangle 8 in. long and 5 in. wide.

Answer. $A =$ ------

7. (65) What kind of solid would be formed if the accompanying figure were cut out and folded along the dotted lines?

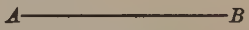
Answer -----------



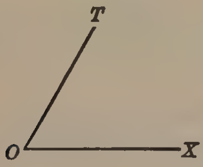
8. (66) Find the following product by an algebraic method:

$$(r + t)(2 + x) =$$

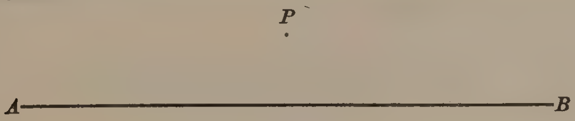
9. (67) Bisect the line segment AB.



10. (68) Bisect the angle XOT.



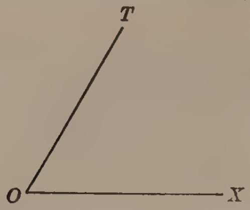
11. (69) Construct a perpendicular to the given line AB from the external point P.



12. (70) Give the meaning of the following expression:

$3\,a$ means ----------------------------------

13. (71) On the working line AB construct an angle equal to the angle XOT by using only the ruler and compasses.



14. (72) Draw a figure to represent the following product:

$$(a + b)(c + d).$$

15. (73) Find the volume of a cube whose edge is 2.2. cm.

Answer. $V =$ ------

16. (74) What is the total surface of a cube one of whose edges is 2?

Answer -----------

17. (75) Draw a figure to represent $(m + n)^2$.

18. (76) Give the meaning of the following expression:

$3\,a^2$ means ------------------------------------

19. (77) Solve for x in the following equation:

$3(x + 15) + 5 = 2(2\,x + 9) + 4(x + 3)$. *Answer.* $x =$ -------

20. (78) Draw an acute triangle ABC and construct the perpendicular bisectors of the three sides. Leave all the construction marks.

21. (79) Draw an acute triangle ABC and construct the bisectors of the three angles of the triangle.

22. (80) Construct a perpendicular to the line AB at the point B.

23. (81) Fill in the blanks with numbers and words that will make a correct statement:

A cube has -------- faces, all of which are ---------. Two faces intersect in an ---------. A cube has -------- edges and ------- corners. Each corner is formed by the intersection of ---------.

24. (82) How many feet of wire fencing are needed to inclose a square whose area is $9\,a^2$ sq. ft.? *Answer*___________

25. (83) Draw an acute triangle *ABC* and construct the three altitudes of the triangle.

26. (84) Find the following indicated product:

$$(3\,br)(4\,b^2r)(6\,b^3r) =$$

27. (85) Construct a square whose side is 3 cm. long.

28. (86) Two parallel lines are cut by a transversal so as to form two corresponding angles $(3x - 150)^\circ$ and $(x + 80)^\circ$. Find x and the size of each angle. *Answer*___________

29. (87) What is the volume of a parallelepiped whose edges are x, x, and $3\,x + 2$? *Answer*___________

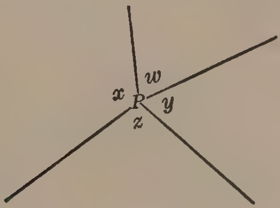
30. (88) A rectangular box containing c cubic inches is a inches wide and 6 in. deep. How long is the box? *Answer*___________

GENERAL-MATHEMATICS SCALE

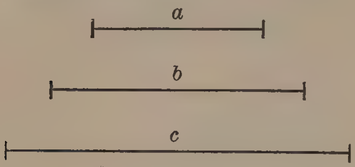
SERIES C, FORM I

Fundamental Angle Relations; the Triangle; Addition and Subtraction of Directed Numbers

1. (58) Express by means of an equation the sum of the angles in the following figure:



2. (61) On the working line below construct a triangle, using the three given sides a, b, and c.



3. (62) A ship starting in latitude $+20°$ sails on the same meridian $+17°$ and then $-18°$. What is its latitude after these sailings?

*Answer*___________

4. (63) Add:

$$\begin{array}{r} 2x+4y+3z \\ -5x-7y-6z \\ 4x+8y+5z \\ 2x-2y-5z \\ \hline \end{array}$$

5. (64) Add $+40, -10, -30, +18, -23, -31, +16$.

*Answer*___________

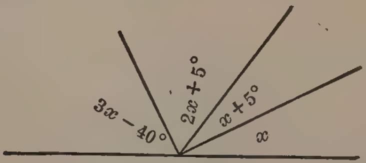
6. (67) The first angle of a triangle is twice the second, and the third is three times the first. Find the number of degrees in each angle.

Answers __________

7. (68) A newsboy who has \$30 in the bank deposits \$10.50 on Monday, checks out \$13.41 on Tuesday, deposits \$5.72 on Wednesday, checks out \$7.18 on Thursday, checks out \$12.17 on Friday, and deposits \$9.75 on Saturday. What is his balance at the end of the week?

*Answer*___________

8. (70) Express by means of an equation the sum of the angles in the following figure. Solve the equation and find the number of degrees in each angle.



Answers __________

9. (71) Perform the operations indicated in the following case and simplify the result as much as possible:

$$7x + (-3x) =$$

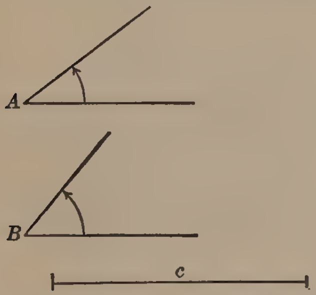
10. (71) Subtract:

$$\begin{array}{r} 7(a+b) \\ \underline{-8(a+b)} \end{array}$$

11. (71) Subtract:

$$\begin{array}{r} +8a - 12b - 156 \\ \underline{-5a + 7b - 75} \end{array}$$

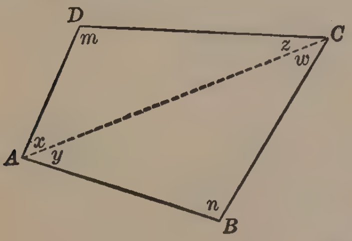
12. (73) On the working line below construct a triangle, using the two given angles and the included side *c*:



13. (74) The acute angles of a right triangle are $\frac{x}{2}$ degrees and $\frac{x}{3}$ degrees. Find x and the number of degrees in each angle.

*Answer*___________

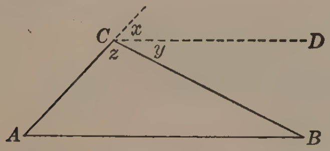
14. (74) Using the accompanying figure, show in the space at the right that the sum of the interior angles of a quadrilateral is 360°.



15. (75) Subtract:

$$\begin{array}{r} -ab + a^2 - 3b^2 \\ \underline{+ab - a^2 + 2b^2} \end{array}$$

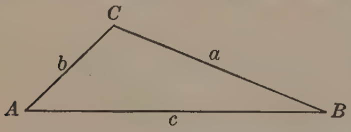
16. (76) In the figure below, CD is drawn parallel to AB. Show in the space at the right that the sum of the interior angles of the triangle ABC is 180°.



17. (77) A pair of vertical angles have respectively the values $\left(\frac{y}{3}+\frac{y}{6}\right)^\circ$ and $\left(\frac{y}{4}+18\right)^\circ$. Find the size of each angle.

Answer ___________

18. (77) Measure each angle of triangle ABC with a protractor, and fill in the blanks to indicate the number of degrees in each angle.



$$\textit{Answers.}\begin{cases}\text{Angle } A = \_\_\_ \text{ degrees.}\\ \text{Angle } B = \_\_\_ \text{ degrees.}\\ \text{Angle } C = \_\_\_ \text{ degrees.}\end{cases}$$

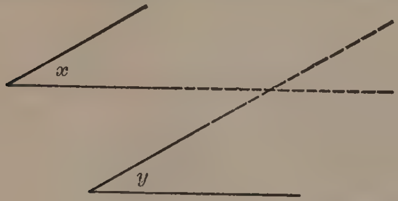
19. (78) In the following case perform the indicated operations and simplify the result as much as possible:

$$7y - (-7 + 2y) =$$

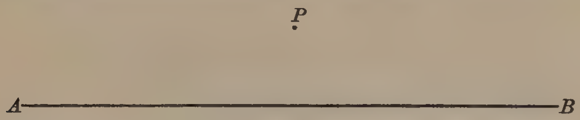
20. (79) Solve for x:

$\frac{1}{3}(x + 4) = 6.$ *Answer.* $x =$ _______

21. (80) In the accompanying figure, angles x and y have their sides respectively parallel. Show in the space at the right that x equals y.



22. (80) Through the point P outside the given line AB construct a line parallel to AB by making the alternate-interior angles equal. Leave all your construction marks on the paper.



23. (82) Solve for x:

$$\frac{5(x+2)}{6}+x=\frac{x}{2}+5.$$

Answer. $x =$ ------

24. (83) In the space below at the right, translate the following subtraction into a problem involving debit and credit:

$$\begin{array}{r} +13 \\ -20 \\ \hline +33 \end{array}$$

25. (84) Perform the indicated operations and simplify the result as much as possible:

$$7x-[5y-(4x+3z)-(7y-6z)]=$$

26. (84) Solve for x:

$$x-3(2-5x)=32.$$

Answer. $x =$ ------

27. (88) Write an algebraic expression which will be a translation of the following statement:

"15° plus $\frac{1}{2}$ of the complement of an angle."

Answer -----------

28. (88) Write but do not solve the equation which expresses the conditions in the following problem:

If you think of an angle, double it, increase its supplement by 15°, and get a result of 250°, what are the two supplementary angles?

Equation ----------

29. (91) In the following case perform the indicated operations and simplify the result as much as possible:

$$\frac{7a+2}{12a}-\frac{4a-3}{4a}=$$

30. (95) In the following case perform the indicated operations and simplify the result as much as possible:

$$5 - \frac{2 - 3a}{5} - 4a =$$

GENERAL-MATHEMATICS SCALE

SERIES D, FORM I

Multiplication and Division of Directed Numbers; Factoring

1. (59) Find the prime factors of the following expression:

$$a^2 - 81 =$$

2. (60) In the following case perform the indicated operations and simplify the result:

$$9(-5x - 2) =$$

3. (61) In the following case perform the indicated operations and simplify the result:

$$-4(6x + 2) =$$

4. (62) Find the prime factors of the following expression:

$$b^2 + 3b - 70 =$$

5. (63) Find the prime factors of the following expression:

$$y^2 - 10y + 25 =$$

6. (64) Find the prime factors of the following expression:

$$x^2 - 1 =$$

7. (65) Find the prime factors of the following expression:

$$x^2 - 6x - 27 =$$

8. (66) Reduce the following fraction to lowest terms:

$$\frac{-14a^7b^3y^8}{-7a^7b^2y^6} =$$

9. (67) Perform the indicated operation and simplify the result in the following expression:

$$(2a + 3b)^2 =$$

10. (68) Perform the indicated operation and simplify the result in the following expression:

$$(2x - 4y)(2x + 4y) =$$

11. (69) Perform the indicated operation and simplify the result in the following expression:

$$x^2 \cdot (-2x) \cdot (-4x) =$$

12. (70) Reduce to lowest terms the following fraction:

$$\frac{+12(x+y)^5}{-2(x+y)^3} =$$

13. (71) Find the prime factors of the following expression:

$$6a^2 - 19a + 15 =$$

14. (72) In the following expression perform the indicated operation:

$$(3a - 2b)^2 =$$

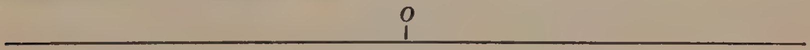
15. (73) In the following expression perform the indicated operation:

$$(a - b)(a^2 + ab + b^2) =$$

16. (74) Find the prime factors of the following expression:

$$15 - 4b - 4b^2 =$$

17. (75) Using *O* on the line below as a starting point, show geometrically that $(-4)(+3) = -12$.



18. (76) Illustrate the product $(a + b)(3a + 2b)$ by means of a geometric figure.

19. (77) In the following expression perform the indicated operations and simplify the result:

$$(-2)^2(-3)^2(-4)(2)^2 =$$

20. (78) Find the quotient indicated below:

$$\frac{x^3 + x^2 + x}{x} =$$

21. (79) Find the quotient indicated below:

$$\frac{12m^3n^4 - 2m^6n^5r}{-6m^2n^3} =$$

22. (80) In the following expression perform the indicated operation:

$$(2a - 3b + c)^2 =$$

23. (81) Find the quotient indicated below:

$$\frac{-15x^3y^3(y-3)}{-5x^2y^2} =$$

24. (82) Find the prime factors of the following expression:

$$4a^2 + 16a + 16 =$$

25. (83) Find the quotient indicated below:

$$\frac{8x}{12} \div \frac{9x^3}{15} =$$

26. (84) In the following problem perform the indicated operations and simplify the result:

$$\frac{a}{x^2 - y^2} \cdot \frac{x + y}{b} =$$

27. (85) Find the prime factors of the following expression:

$$x^4 - y^4 =$$

28. (86) Find the prime factors of the following expression:

$$8a^2 - 2 =$$

29. (87) Find the prime factors of the following expression:

$$2x^2z + 11xz + 12z =$$

30. (88) In the following expression perform the indicated operations and simplify the result:

$$2x + 2(3x - 2y) - 4(x + 3y) =$$

GENERAL-MATHEMATICS SCALE

SERIES E, FORM I

Statistics; the Graph; the Formula

1. (59) The formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is the number of units of length in the base and h is the number of units of length in the height. Find the number of square feet in the area of a triangle whose base is 20 ft. and whose altitude is 6 ft.

*Answer*___________

2. (59) In the formula $V = \frac{lwh}{3}$, solve for h.

Answer. $h =$ _______

3. (59) In the formula $C = 2\pi r$, solve for r.

Answer. $r =$ _______

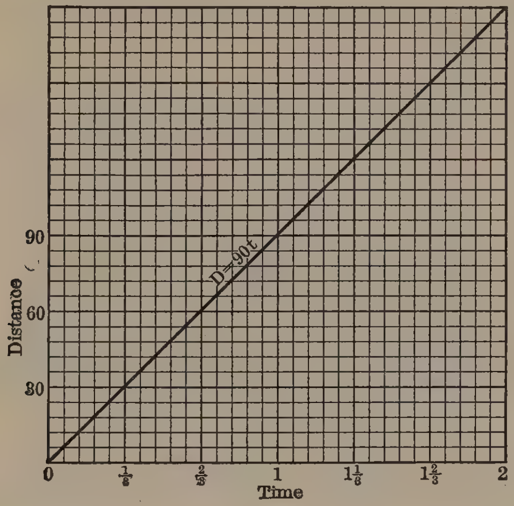
4. (60) In the formula $A = \frac{Pl}{2}$, solve for l.

Answer. $l =$ _______

5. (67) In the formula $w_1l_1 = w_2l_2$, solve for l_2.

Answer. $l_2 =$_______

6. (68) In the following diagram the graph represents the distance traveled by an automobile whose rate was 90 mi. per hour over a period of hours:



From the graph and the relation $D = 90\,t$ we see that

$$\text{if } t = \tfrac{1}{3}, \text{ then } D = 30;$$
$$\text{if } t = \tfrac{2}{3}, \text{ then } D = 60;$$
and $$\text{if } t = 1, \text{ then } D = 90;$$

and so on for any other values of t.

On the same section of squared paper draw a graph showing the distance passed over by an automobile whose rate is 60 mi. per hour; that is, where $D = 60\,t$.

7. (68) In the formula $PV = K$, solve for P.

Answer. $P =$ -------

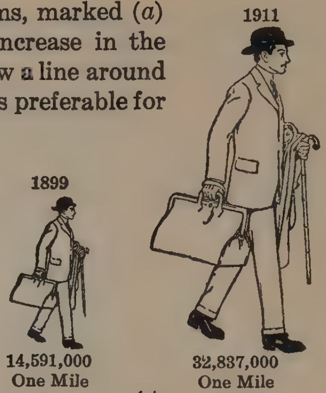
8. (70) In the formula $C = \frac{5}{9}(F - 32)$, solve for F.

Answer. $F =$ -------

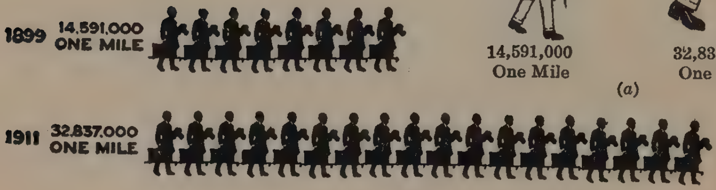
9. (72) The area of a circle is given by the formula $A = \pi r^2$, where r is the number of units of length in the radius and $\pi = 3\frac{1}{7}$. Find the area in square feet of a circle whose radius is 9 ft.

Answer -----------

10. (73) The two accompanying diagrams, marked (*a*) and (*b*), are each intended to show the increase in the passenger traffic of a certain railroad. Draw a line around the one of the two types which you think is preferable for representing the facts.



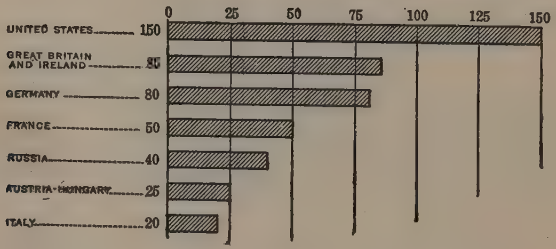
(*a*)



(*b*)

11. (73) The following bar diagram represents the estimated wealth, in billions of dollars, of certain nations in 1914. How many billions of dollars is France represented as being worth in this diagram?

Answer ___________



12. (74) Write but do not solve the equation which expresses the conditions in the following problem:

The distance from Minneapolis to Chicago is 421 mi. If a train runs at the rate of r miles an hour, what time is required for the run?

Equation __________

13. (74) In the formula $q = \frac{D}{d}$, solve for d. *Answer.* $d =$ _______

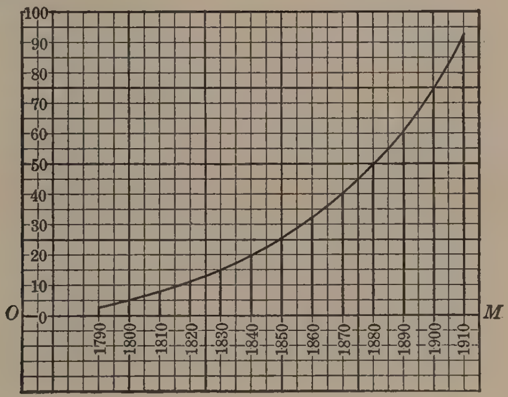
14. (74) In the formula $C = \frac{5}{9}(F - 32)$, solve for C when $F = 60°$.

Answer. $C =$ _______

15. (75) In the formula $A = P + Prt$, solve for t.

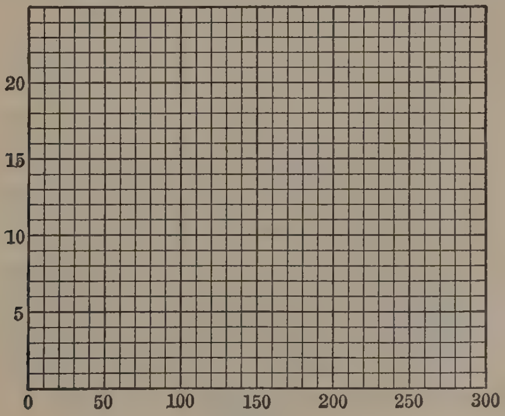
Answer. $t =$ _______

16. (76) The following graph represents the population, in millions, of the United States from 1790 to 1910. What was the probable population in 1885?



Answer ___________

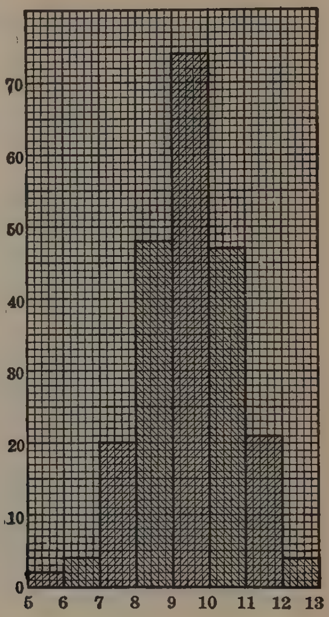
17. (77) Using the accompanying section of squared paper, draw a price graph to represent the cost of eggs at 70¢ per dozen.



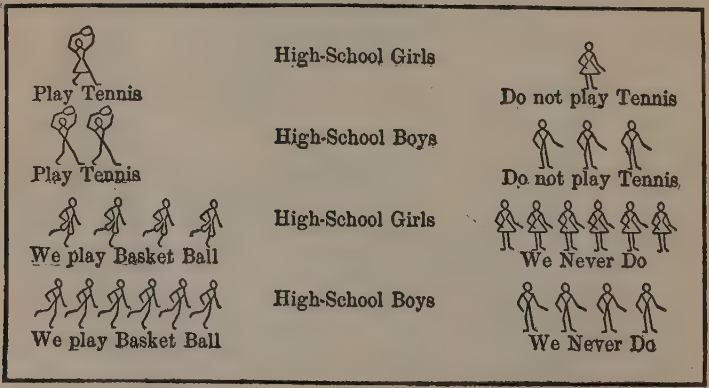
18. (78) The accompanying diagram is a frequency polygon showing the distribution of 220 ears of corn with respect to lengths in inches.

1. How many ears of corn are there measuring somewhere between 10 in. and 11 in. in length? *Answer*____________

2. What is the most common length of ear? *Answer*____________



19. (79) The accompanying pictograms show the tennis and basketball interests of the boys and girls in a certain high school. From the diagram fill in the blanks in the statement on the next page:



Per cent of girls playing tennis --------------------------------
Per cent of boys playing tennis --------------------------------
Per cent of girls playing basket ball ----------------------------
Per cent of boys playing basket ball ----------------------------
Tennis is more popular with the ------------------------------

20. (79) In the formula $S = \frac{1}{2} gt^2$, solve for t.

Answer. $t =$ -------

21. (80) In the formula $S = 2\,rh + 2\,\pi r^2$, solve for h.

Answer. $h =$ -------

22. (80) Write but do not solve the equation which expresses the conditions in the following problem:

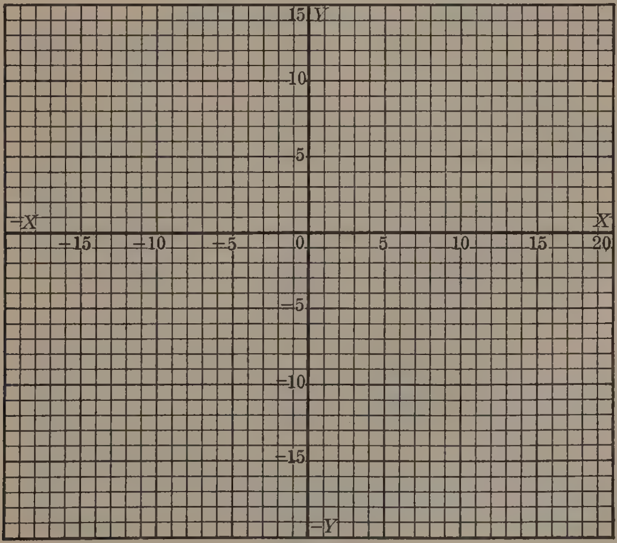
A man invests $6000 in two parts. The first part he invests at 4% and the second part at 5%. If the total annual income from the two investments is $280, what is the amount of each?

Equation ----------

23. (82) In the formula $V = \frac{4}{3}\pi r^3$, solve for r.

Answer. $r =$ -------

24. (83) Select three pairs of corresponding values for x and y in the equation $2\,x - y = 6$ and draw its graph on this section of square paper:



25. (83) In the formula $L = \frac{MX - g}{X}$, solve for M.

Answer. $M =$ ------

26. (85) In the formula $C = \frac{E}{R + r}$, solve for r.

Answer. $r =$ -------

27. (88) The volume of a sphere is the cube of its radius multiplied by $\frac{4}{3}\pi$. Find V when $r = 12$ ft.

Answer. $V =$ ------

28. (89) Write but do not solve the equation which expresses the conditions in the following problem:

A train leaves Chicago at the rate of 50 mi. an hour. Three hours later another train leaves the same station over the same road and travels in the same direction at the rate of 60 mi. an hour. How far from Chicago will the second train overtake the first?

Equation ----------

29. (93) In the formula $S = \frac{a}{1 - r}$, solve for r.

Answer. $r =$ -------

30. (93) Using the facts of the following table showing church contributions, fill in the blanks below the table:

INDIVIDUAL CONTRIBUTIONS	NUMBER OF CONTRIBUTORS
0	2
1 cent	23
5 cents	42
10 cents	13
25 cents	9
$50 check	1

The arithmetic average is ------------------------------
The mode is --
The median is --

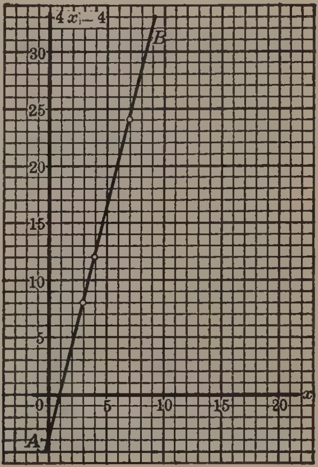
GENERAL–MATHEMATICS SCALE

SERIES F, FORM I

The Function; Similarity and Proportion; Trigonometry

1. (62) The accompanying diagram represents a graph of the function $4x - 4$. Pick out, from the graph, the values of x that will make the function $4x - 4$ have the following values:

1. If $4x - 4 = 0$, then $x =$ ------.
2. If $4x - 4 = 12$, then $x =$ ------.
3. If $4x - 4 = 28$, then $x =$ ------.



2. (63) In the parentheses below underline the word that makes a true statement:

As an angle A increases from 0° to 90° cosine A (increases, decreases).

3. (64) Complete the statement:

A constant is ----------------------------------

4. (65) State a quantity upon which the following depends:

The distance that a man drives his car in 12 hr.

Answer -----------

5. (66) Complete the following statement:

A variable is ----------------------------------

6. (67) Write but do not solve the equation which expresses the conditions in the following problem:

From a point A on level ground the angle of elevation of an airplane is 55°. A point C on the ground directly beneath the airplane is 450 yd. from A. Find the height of the airplane. *Equation* ----------

7. (69) The simple interest on an investment varies directly as the time. If the interest for 4 yr. on a sum of money is \$200, what will be the interest for 6 yr.?

*Answer*___________

8. (72) Why is the sine of a given acute angle a function of the angle?

*Answer*___________

9. (73) In the formula $C = \pi d$

___________is the dependent variable

and ___________is the independent variable.

10. (74) State a quantity upon which the following depends: The cost of a sirloin steak.

*Answer*___________

11. (74) If x varies directly with y, and $x = 12$ when $y = 3$, find x when $y = 7$.

Answer. $x =$_______

12. (75) If $PV = K$ is the formula which says that the pressure of gas varies inversely as the volume, and the pressure of a gas is 4 kg. when the volume is 14 cc., what is the volume when the pressure is 7 kg.?

*Answer*___________

13. (76) Give an example of direct variation and an example of inverse variation:

Direct variation ___________________________________

Inverse variation ___________________________________

14. (76) If y varies inversely with x, and $y = 8$ when $x = 12$, find y when $x = 4$.

Answer. $y =$_______

15. (77) By underlining the proper word in each case below, show which are examples of direct variation and which are examples of inverse variation:

1. $\frac{x}{y} = k$ (direct, inverse).
2. $xy = k$ (direct, inverse).
3. $C = \pi d$ (direct, inverse).
4. $PV = K$ (direct, inverse).
5. $V = \frac{4}{3}\pi r^3$ (direct, inverse).

16. (78) In the formula $C = \frac{5}{9}(F - 32)$,

___________is the dependent variable

and ___________is the independent variable.

17. (79) Sketch the figure and solve the right triangle ABC for the unknown parts when $A = 30^\circ$ and $a = 35$ ft.

18. (80) If 200 ft. of copper wire weighs 60 lb., what is the weight of 125 ft. of the same kind of wire? *Answer*----------

19. (80) Write but do not solve the equation which expresses the conditions in the following problem:

A kite which is 200 yd. above the ground makes an angle of elevation of 46° with a point A where the string is fastened. Find the length of the string. *Equation*----------

20. (80) Sketch the figure and solve the right triangle ABC for the unknown parts when $A = 40^\circ$ and $c = 75$ ft.

21. (84) Sketch the figure and solve the right triangle ABC for the unknown parts when $b = 90$ ft. and $c = 100$ ft.

22. (85) Write but do not solve the equation which expresses the conditions in the following problem:

A and B together weigh 220 lb. They balance when A is 5 ft. and B is 6 ft. from the fulcrum. Find the weight of each.

Equation----------

23. (86) Write but do not solve the equation which expresses the conditions in the following problem:

The shadow of a chimney is 80 ft. long. At the same time the shadow of a man 6 ft. 1 in. tall is 9 ft. 2 in. long. How high is the chimney? *Equation*----------

24. (86) Write but do not solve the equation which expresses the conditions in the following problem:

The area of a rectangle is 48 sq. ft. and the base is 33 ft. What is the area of a rectangle having the same altitude and a base equal to 72 ft.? *Equation*----------

25. (87) On the working line below construct the mean proportional between the two given line segments a and b.

a

b

26. (88) On the working line below construct the fourth proportional to the three given line segments a, b, and c.

a

b

c

27. (88) Sketch the figure and solve the right triangle ABC for the unknown parts when $a = 30$ ft. and $b = 80$ ft.

28. (89) Solve for x: $\frac{5}{2 - x} = \frac{3}{4 + x}$. *Answer.* $x =$ -------

29. (90) In a map drawn to the scale of 1 to 100,000, what lengths will represent the boundaries of a rectangular-shaped county 50 mi. long and 30 mi. wide? Give the results to the nearest hundredth of an inch.

Answer -----------

30. (91) Solve for x: $\frac{x + 4}{x - 5} = \frac{2 + x}{x - 3}$. *Answer.* $x =$ -------

GENERAL-MATHEMATICS SCALE

SERIES G, FORM I

Simultaneous Linear Equations; Quadratic Equations

1. (61) Solve the following set of equations:

$$3x + 2y = 7,$$
$$2x + 3y = 8.$$

Answer. $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

2. (62) Solve the following quadratic equation by factoring:

$x^2 - 64 = 0$. *Answer.* $x =$ -------

3. (63) Solve the following set of equations:

$$3a + 4b = 11,$$
$$5a - b = 3.$$

Answer. $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

4. (66) Solve the following quadratic equation by factoring:

$x^2 + 6x = 7$. *Answer.* $x =$ -------

5. (67) Solve the following quadratic equation by factoring:

$x^2 - 5x = 50$. *Answer.* $x =$ -------

6. (67) Solve the following quadratic equation by factoring:

$x^2 + 7x = 8$. *Answer.* $x =$ -------

7. (68) Solve the following quadratic equation by factoring:

$x^2 + 5x + 6 = 0$. *Answer.* $x =$ -------

8. (69) Solve the following set of equations:

$$2x = 3y + 1, \qquad x + y = 8.$$

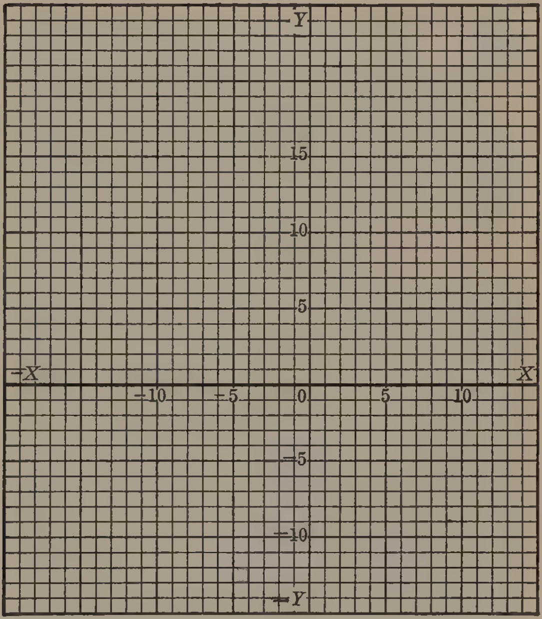
Answer. $\left\{\begin{array}{l} x = \text{------} \\ y = \text{------} \end{array}\right.$

9. (69) Solve the following quadratic equation by factoring:

$x^2 - 5x + 6 = 0.$ *Answer.* $x =$ -------

10. (72) Solve $x^2 - 7x + 6 = 0$ graphically, using the table started below.

x	$x^2 - 7x + 6$
-1	
0	
1	
2	
3	
4	
5	
6	
7	



11. (73) Solve the following quadratic equation by factoring:

$z^2 - 85 = 12z.$ *Answer.* $z =$ -------

12. (74) Solve the following set of equations:

$$\frac{2}{x} + \frac{3}{y} = \frac{9}{10}, \qquad \frac{3}{x} + \frac{2}{y} = \frac{14}{15}.$$

Answer. $\left\{\begin{array}{l} x = \text{------} \\ y = \text{------} \end{array}\right.$

13. (74) Solve the following quadratic equation by the method of completing the square: $x^2 - 6x = 72.$ *Answer.* $x =$ -------

14. (75) A ladder 20 ft. long reaches a window 16 ft. above the ground. Assuming that the ground is level, how far is the foot of the ladder from the foot of the wall? *Answer*__________

15. (76) Solve the following quadratic equation by factoring:

$20w - 51 = w^2$. *Answer.* $w =$ ________

16. (76) Write but do not solve the equations which express the conditions of the following problem:

Find two numbers such that three times the first plus 5 equals 35, and two times the first plus four times the second equals 68.

Equations { ________
____

17. (77) In the following expression perform the indicated operations and reduce to simplest terms:

$$3\sqrt{72} - 2\sqrt{18} + \sqrt{32} =$$

18. (79) Solve the following set of equations:

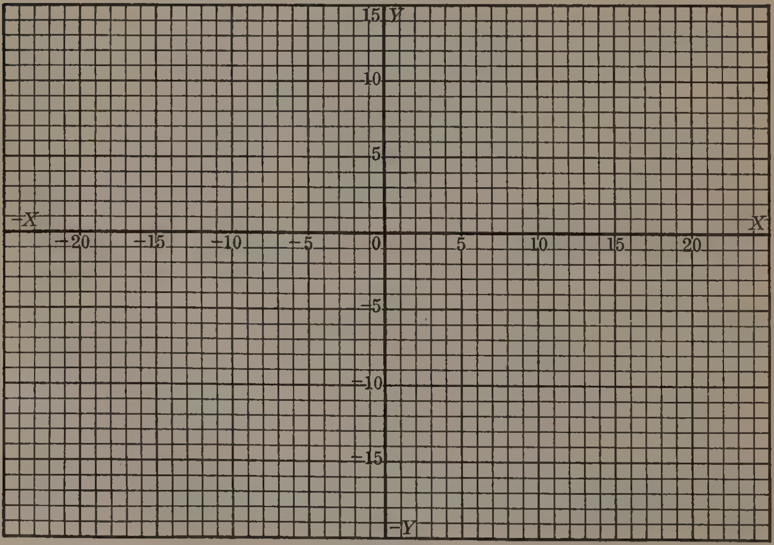
$$3x - 2y = 54,$$
$$\frac{x}{2} - \frac{y}{3} = 9.$$

Answer { $x =$ ______
$y =$ ______

19. (80) Graph the following pair of equations on the section of squared paper below and indicate the solution if they have one:

$$x + y = 6,$$
$$2x - 3y = 2.$$

Answer { $x =$ ______
$y =$ ______



20. (81) Solve the following set of equations:

$$\frac{x}{5}+\frac{y}{2}=4,\qquad \frac{x}{3}-\frac{y}{2}=12.$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

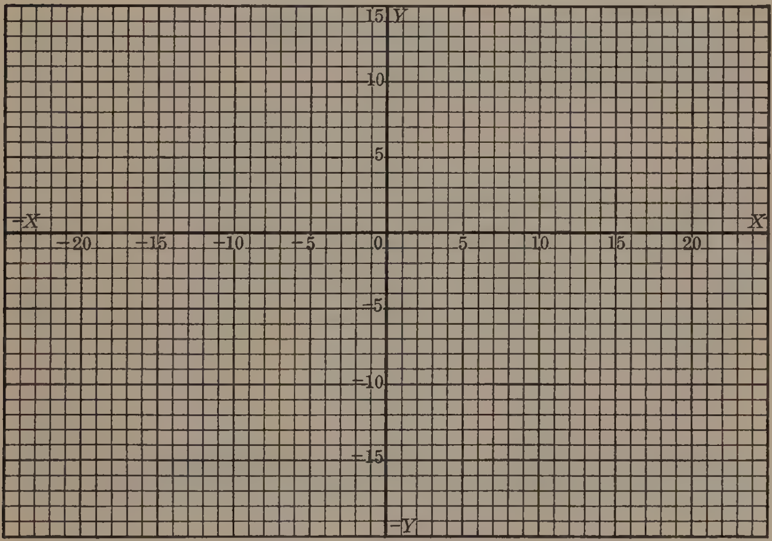
21. (82) In the following expression perform the indicated operations and reduce to the simplest form:

$$\sqrt{12}+\sqrt{48}+\sqrt{\tfrac{1}{3}}=$$

22. (82) Graph the following set of equations on the section of squared paper below and indicate the solution, if they have one:

$$4\,x+3\,y=2,\qquad 8\,x+6\,y=10.$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$



23. (83) Solve the following quadratic equation by the method of completing the square:

$$x^2+4\,x=18.$$

Answer. $x = \text{-------}$

24. (83) Write but do not solve the equation which expresses the conditions in the following problem:

Two trains are 180 mi. apart by rail on two roads that run perpendicular to each other, and both trains are approaching the crossing. One train runs 6 mi. per hour faster than the other. At what rates must they run if they both reach the crossing in 6 hr.? *Equation*----------

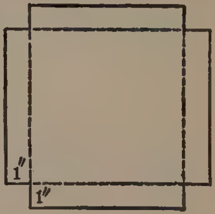
25. (85) Write but do not solve the equations which express the conditions in the following problem:

The perimeter of a football field is 320 yd., and the length is 10 yd. more than twice the width. What are its dimensions?

Equations $\begin{cases} \text{-----------------} \\ \text{-----------------} \end{cases}$

26. (86) Write but do not solve the equation which expresses the conditions in the following problem:

A piece of tin in the form of a square is used to make an open box. The box is made by cutting a 1-inch square from each corner of the piece of tin and folding up the sides. The box thus made contains 56 cu. in. Find the length of the side of the original piece of tin.



Equation----------

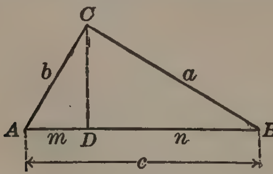
27. (87) In the following expression perform the indicated operation and reduce to the simplest form:

$$\sqrt{x^4 + 2x^3 + 7x^2 + 6x + 9} =$$

28. (88) In the following expression perform the indicated operations and reduce to the simplest form:

$$2a^{\frac{1}{2}} \cdot 5a^{\frac{3}{2}} =$$

29. (91) In the accompanying diagram ABC is a right triangle, right-angled at C. Show in the space at the right that $c^2 = a^2 + b^2$.



30. (98) Solve the following quadratic equation:

$$\frac{4y}{y-1} = 4 - \frac{y-10}{y}.$$

Answer. $y =$-------

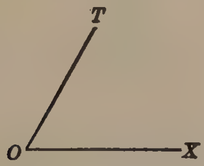
GENERAL-MATHEMATICS COMPOSITE SCALE

FORM I

1. (64) A vessel starting in latitude + 20° sails on the same meridian + 17° and then − 18°. What is its latitude after these sailings?

Answer-----------

2. (65) Bisect the angle *XOT*.

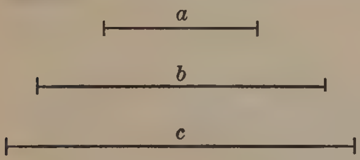


3. (66) Solve the following equation:

$$\frac{y}{3} + 2 = 5.$$

Answer. $y =$-------

4. (67) Construct a triangle, using the three sides *a*, *b*, and *c* given below.



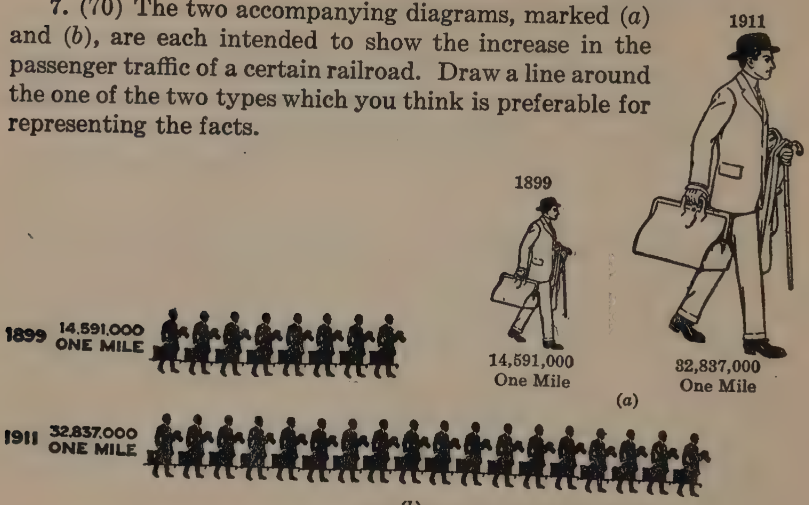
5. (68) Add:

$$\begin{array}{r} 2x + 4y + 3z \\ -5x - 7y - 6z \\ 4x + 8y + 5z \\ \underline{2x - 2y - 5z} \end{array}$$

6. (69) Find the prime factors of the following expression:

$$a^2 - 81 =$$

7. (70) The two accompanying diagrams, marked (*a*) and (*b*), are each intended to show the increase in the passenger traffic of a certain railroad. Draw a line around the one of the two types which you think is preferable for representing the facts.



(*a*)

(*b*)

8. (71) Give the meaning of the following expression:

a^3 means --

9. (72) Find the volume of a cube whose edge is 2.2 cm.

Answer. $V =$ ------

10. (73) The area of a circle is given by the formula $A = \pi r^2$, where r is the number of units of length in the radius and $\pi = 3\frac{1}{7}$. Find the area in square feet of a circle whose radius is 9 ft.

Answer. $A =$ ------

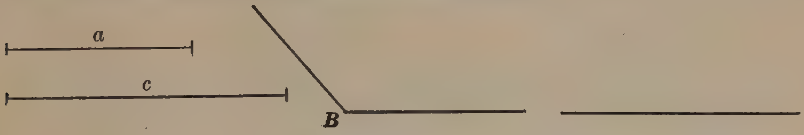
11. (74) In the formula $C = \frac{5}{9}(F - 32)$, solve for C when $F = 60°$.

Answer. $C =$ ------

12. (75) In the formula $C = \frac{5}{9}(F - 32)$

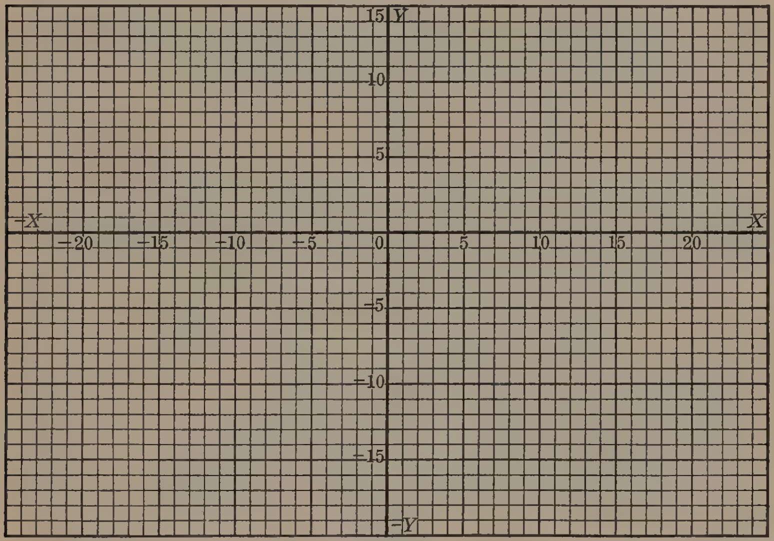
----------- is the dependent variable
and ----------- is the independent variable.

13. (76) On the working line below construct a triangle, using the two given sides a and c and the included angle B.

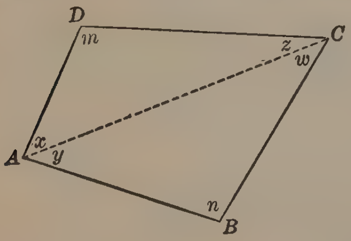


14. (76) Select three pairs of corresponding values for x and y in the following equation and draw its graph on the section of squared paper given below:

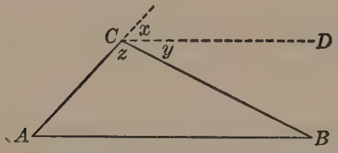
$$2x - y = 6.$$



15. (78) Using the accompanying figure, show in the space at the right that the sum of the interior angles of a quadrilateral is 360°.



16. (79) In the figure below, CD is drawn parallel to AB. Show in the space at the right that the sum of the interior angles of the triangle ABC is 180°.



17. (80) Write but do not solve the equation which expresses the conditions in the following problem:

The shadow of a chimney is 80 ft. long. At the same time the shadow of a man 6 ft. 1 in. tall is 9 ft. 2 in. long. How high is the chimney?

Equation ----------

18. (81) Write but do not solve the equations which express the conditions in the following problem:

Find two numbers such that three times the first plus 5 equals 35, and two times the first plus four times the second equals 68.

Equations { -----------------
----

19. (82) Two parallel lines are cut by a transversal so as to form two corresponding angles of $(3x - 150)^\circ$ and $(x + 80)^\circ$. Find x and the size of each angle.

Answers ----------

20. (83) Find the prime factors of the following expression:

$$9y + 22y^2 - 31 =$$

21. (84) A ladder 20 ft. long reaches a window 16 ft. above the ground. Assuming that the ground is level, how far is the foot of the ladder from the foot of the wall?

Answer ----------

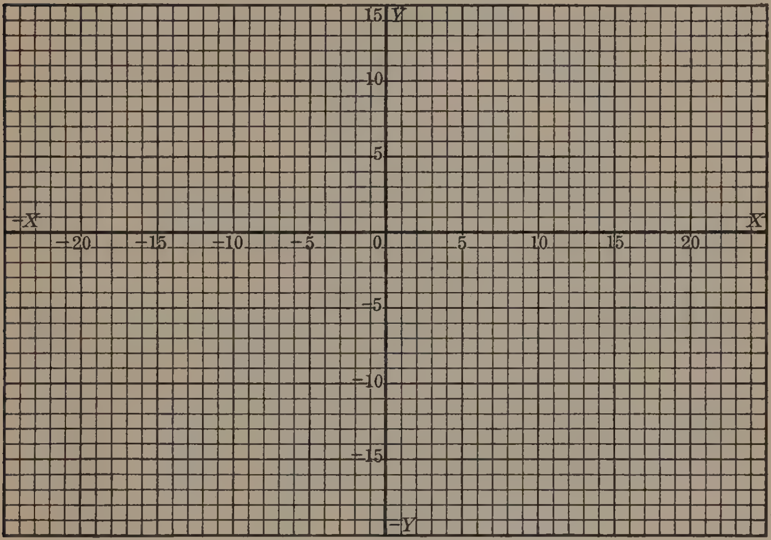
22. (85) State a quantity upon which the following depends:

The height of a ten-year-old boy. *Answer* ----------

23. (86) Graph the following pair of equations on the section of squared paper given below. If there is a solution, indicate what it is.

$$\begin{aligned} 4x + 3y &= 2, \\ 8x + 6y &= 10. \end{aligned}$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$



24. (87) Draw an acute triangle *ABC* and construct the altitudes of the triangle.

25. (88) Using *O* on the line below as a starting point, show geometrically that $(-4)(+3) = -12$.

26. (89) By underlining the proper word in each case, show which of the following are examples of direct variation and which are examples of inverse variation:

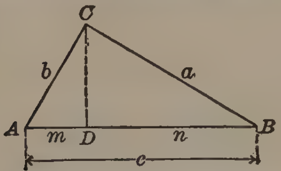
1. $\frac{x}{y} = k$ (direct, inverse).
2. $xy = k$ (direct, inverse).
3. $C = \pi d$ (direct, inverse).
4. $PV = K$ (direct, inverse).
5. $V = \frac{4}{3}\pi r^3$ (direct, inverse).

27. (90) Solve:

$$\frac{2}{x}+\frac{3}{y}=\frac{9}{10}$$
$$\frac{3}{x}+\frac{2}{y}=\frac{14}{15}$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

28. (91) In the diagram below, ABC is a right triangle, right-angled at C. Show in the space at the right that $c^2 = a^2 + b^2$.



29. (92) Write but do not solve the equations which express the conditions in the following problem:

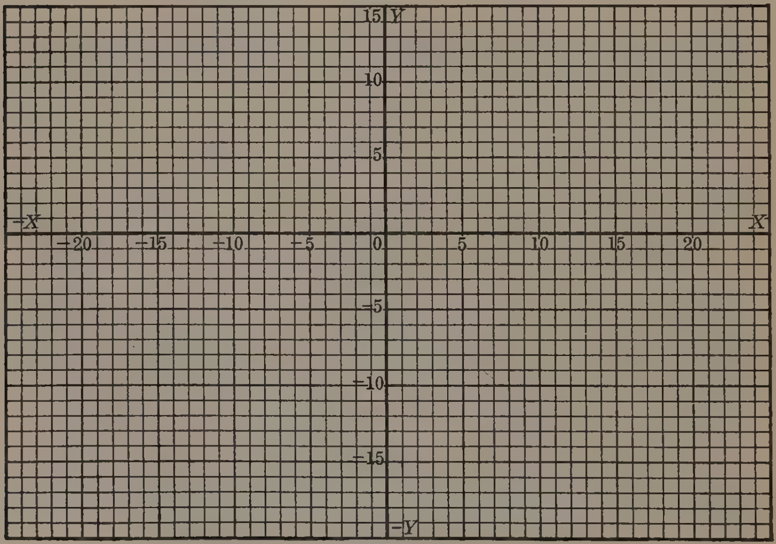
Two grades of coffee costing a grocer 30¢ and 50¢ a pound are to be mixed so that 40 lb. of the mixture will be worth 42¢ a pound. How many pounds of each kind of coffee must be used in the mixture?

Equations---------

30. (93) Graph the following pair of equations on the section of squared paper below, and if there is a solution indicate what it is:

$$x + y = 6,$$
$$2x - 3y = 2.$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$



GENERAL-MATHEMATICS COMPOSITE SCALE

FORM II

1. (64) Find the value of the following expression when $a = 5$:

$4a.$ *Answer* ___________

2. (65) Find the following quotient:

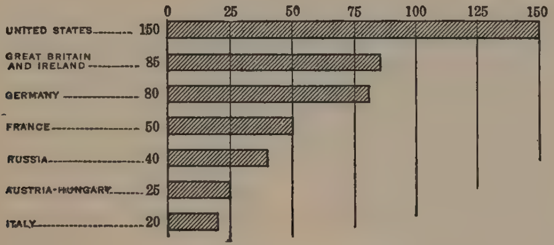
$$\frac{16a^3 - 8a^5}{4a^2} =$$

3. (66) Solve for the value of the unknown in the following equation:

$\frac{y}{3} + 2 = 5.$ *Answer.* $y =$ _______

4. (67) The following bar diagram represents the estimated wealth, in billions of dollars, of certain nations in 1914. How many billions of dollars is France represented as being worth in this diagram?

Answer ___________



5. (68) In the space below at the right, translate the following subtraction into a problem involving debit and credit:

$$\begin{array}{r} +13 \\ \underline{-20} \\ +33 \end{array}$$

6. (69) Find the prime factors of the following expression:

$$a^2 - 81 =$$

7. (70) Subtract:

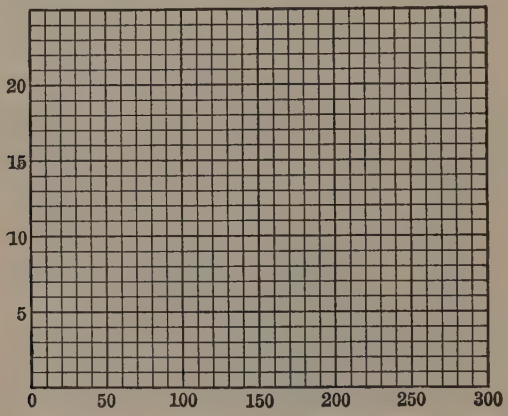
$$\begin{array}{r} -ab + a^2 - 3b^2 \\ \underline{+ab - a^2 + 2b^2} \end{array}$$

8. (71) Write but do not solve the equation which expresses the conditions in the following problem:

The number of representatives and senators together in a certain legislature is 142. The number of representatives is 26 more than the number of senators. Find the number of each. *Equation*----------

9. (72) In the formula $V = Bh$, solve for h. *Answer.* $h =$ -------

10. (73) Using the accompanying section of squared paper, draw a price graph to represent the cost of eggs at 70¢ per dozen.



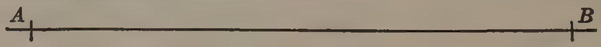
11. (74) In the formula $C = \frac{5}{9}(F - 32)$, solve for C when $F = 60°$.

Answer. $C =$ ------

12. (75) Write but do not solve the equation which expresses the conditions in the following problem:

A man divided his farm of 13,000 acres among his three boys. The first received twice as much as the second and 2000 more acres than the third. How much did each boy receive? *Equation*----------

13. (76) Measure the line AB below and express the result in inches and also in centimeters:



$$\textit{Answers}\begin{cases} AB = \text{----in.} \\ AB = \text{----cm.} \end{cases}$$

14. (77) Solve for the value of the unknowns in the following equations:

$$3x + 2y = 7,$$
$$2x + 3y = 8.$$

Answer $\begin{cases} x = \text{-------} \\ y = \text{-------} \end{cases}$

15. (78) How many feet of wire fencing are needed to inclose a square whose area is 4900 sq. ft.?

Answer ----------

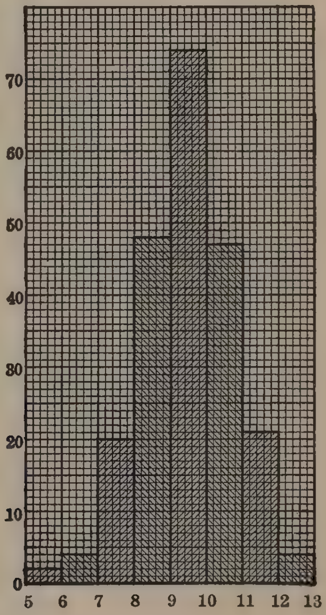
16. (79) The acute angles of a right triangle are $\frac{x}{2}$ degrees and $\frac{x}{3}$ degrees. Find x and the number of degrees in each angle.

Answer -----------

17. (80) The accompanying diagram is a frequency polygon showing the distribution of 220 ears of corn with respect to lengths in inches.

1. How many ears of corn are there measuring somewhere between 10 in. and 11 in. in length? *Answer* -----------

2. What is the most common length of ear? *Answer* -----------

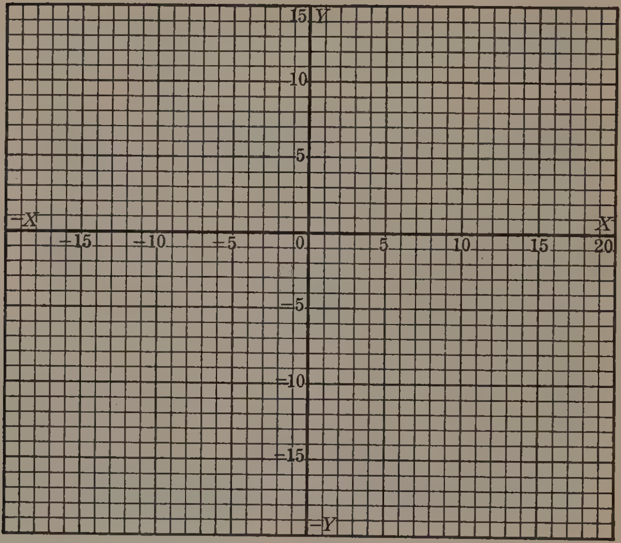


18. (81) In the formula $A = \frac{h(b_1 + b_2)}{2}$, solve for h.

Answer. $h =$ -------

19. (82) Graph the following pair of equations on the section of squared paper below, and if there is a solution indicate what it is:

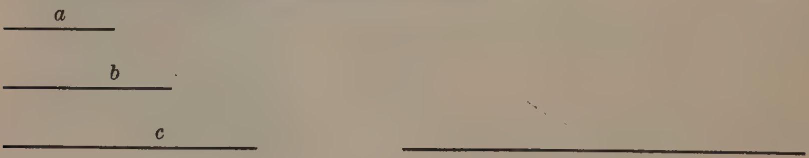
$$x = y + 5,\qquad x + y = 11.\qquad \textit{Answer}\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$$



20. (83) Find the following quotient:

$$\frac{x^3 + x^2 + x}{x} =$$

21. (84) On the working line below construct the fourth proportional to the three given line segments *a*, *b*, and *c*.



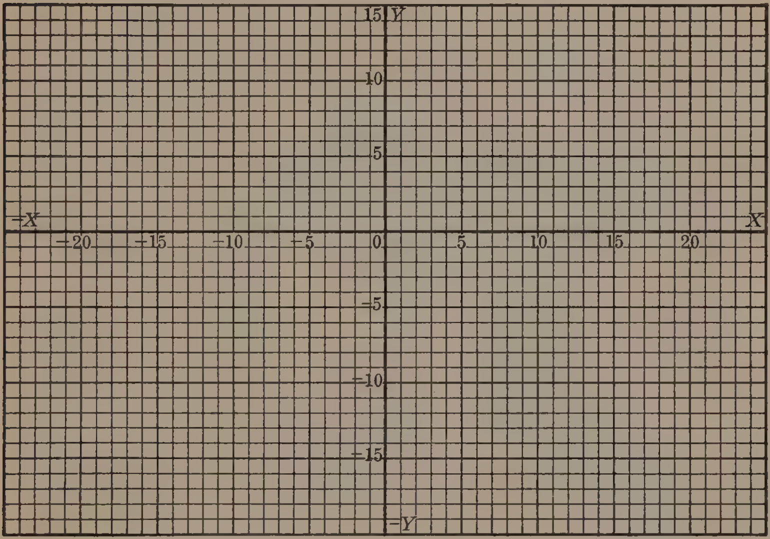
22. (85) Perform the indicated operations in the following expression and simplify the result as much as possible:

$$7x - [5y - (4x + 3z) + (7y - 6z)] =$$

23. (86) Graph the following pair of equations on the section of squared paper given below, and if there is a solution indicate what it is:

$$4x + 3y = 2,$$
$$8x + 6y = 10.$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$



24. (87) Draw an acute triangle ABC and construct the three altitudes of the triangle.

25. (88) Write but do not solve the equation which expresses the conditions in the following problem:

A man invests \$6000 in two parts. The first part he invests at 4% and the second part at 5%. If the total annual income from the two investments is \$280, what is the amount of each?

Equation ----------

26. (89) Find the value of $\frac{3x(x^2 - 5xy + 3y^2)}{x - 3y}$ for $x = 2$ and $y = -3$.

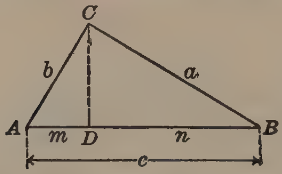
Answer -----------

27. (90) Solve for the values of the unknowns in the following equations:

$$\frac{2}{x} + \frac{3}{y} = \frac{9}{10},$$
$$\frac{3}{x} + \frac{2}{y} = \frac{14}{15}.$$

Answer $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

28. (91) In the diagram below, ABC is a right triangle, right-angled at C. Show in the space at the right that $c^2 = a^2 + b^2$.



29. (92) Using the facts of the following table showing church contributions, fill in the blanks below the table:

INDIVIDUAL CONTRIBUTIONS	NUMBER OF CONTRIBUTORS
0	2
1 cent	23
5 cents	42
10 cents	13
25 cents	9
$50 check	1

The arithmetic average is____
The mode is____
The median is____

30. (93) Write an algebraic expression which will be a translation of the following statement: "15° plus $\frac{1}{2}$ of the complement of an angle."

*Answer*____________

GENERAL-MATHEMATICS COMPOSITE SCALE

FORM III

1. (64) Perform the indicated operations:

$$4x - 6y + 5x - 3y =$$

2. (65) Solve for the value of the unknown in the following equation:

$$2x + 6\tfrac{1}{2} = 12.$$

Answer. $x =$_______

3. (66) Reduce to lowest terms:

$$\frac{-15x^4y^3z^6}{+3x^2yz^5} =$$

4. (67) Write but do not solve the equation which expresses the conditions in the following problem:

Find the sides of a triangle if the second side is 4 ft. longer than the first, the third side 6 ft. longer than the first, and the perimeter 40 ft.

Equation----------

5. (68) Find the value of A if $a = 4$, $b = 1$:

$$A = \frac{4(a+b)}{2}. \qquad \textit{Answer.}\quad A = \text{------}$$

6. (69) Find the prime factors of the following expression:

$$x^2 + 13x + 36 =$$

7. (70) Subtract:

$$\begin{array}{r} +8a - 12b - 156 \\ \underline{-5a + 7b - 75} \end{array}$$

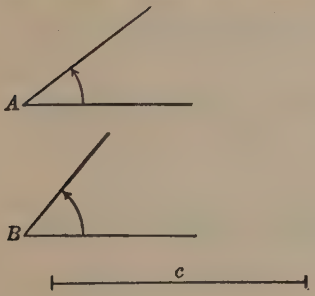
8. (71) Find the prime factors of the following expression:

$$x^2 - 1 =$$

9. (72) Perform the indicated operations and simplify the result in the following expression:

$$(x + y + z)^2 =$$

10. (73) On the working line below construct a triangle, using the two given angles and the included side c:

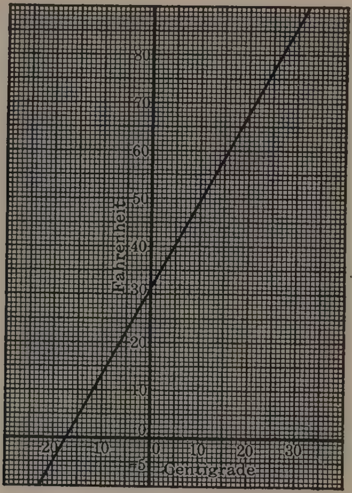


11. (74) A newsboy who has $30 in the bank deposits $10.50 on Monday, checks out $13.41 on Tuesday, deposits $5.72 on Wednesday, checks out $7.18 on Thursday, checks out $12.17 on Friday, and deposits $9.75 on Saturday. What is his balance for the week?

Answer-----------

12. (75) The accompanying graph may be used in changing centigrade readings to Fahrenheit readings, and vice versa. When it is + 10° on the centigrade scale, what is the corresponding reading on the Fahrenheit scale?

Answer-----------



13. (76) Solve for the value of the unknowns in the following equations:

$$2x = 3y + 1,$$
$$x + y = 8.$$

Answer. $\begin{cases} x = \text{------} \\ y = \text{------} \end{cases}$

14. (77) If 200 ft. of copper wire weighs 60 lb., what is the weight of 125 ft. of the same kind of wire?

Answer-----------

15. (78) Solve for the value of the unknown in the following equation:

$$\frac{x}{4} + 4x - \frac{5x}{3} = 49 - 1\tfrac{1}{2}x.$$

Answer. $x =$ -------

16. (79) Write but do not solve the equation which expresses the conditions of the following problem:

Find the angles of a triangle if the first angle is twice the second, and the third is 20° less than twice the first.

Equation----------

17. (80) In the formula $A = \frac{Pl}{2}$, solve for l. *Answer.* $l =$ -------

18. (81) In the formula $q = \frac{D}{d}$, solve for d. *Answer.* $d =$ -------

19. (82) Perform the following indicated division:

$$\frac{3^{10}}{3^5} =$$

20. (83) Draw a figure to represent the following product:

$$b(x + 2\,y + 3).$$

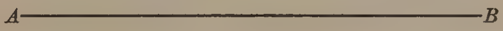
21. (84) Complete the following statement:

A function of a variable is ---------------------------

22. (85) State a quantity upon which the following depends:

The height of a ten-year-old boy. *Answer* -----------

23. (86) Divide the line segment AB into two segments whose ratio is $\frac{3}{4}$:

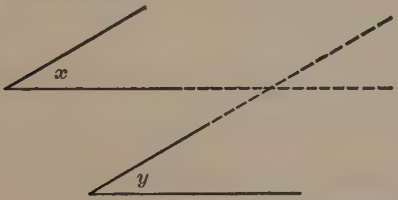


24. (87) The volume of a sphere is the cube of its radius multiplied by $\frac{4}{3}\,\pi$. Find V when $r = 12$ ft. *Answer.* $V =$ -------

25. (88) Find the prime factors of the following expression:

$$8\,a^2 - 2 =$$

26. (89) In the accompanying figure, angles x and y have their sides parallel left to left and right to right. Show in the space at the right that x equals y.



27. (90) Perform the indicated operations and reduce to the simplest form:

$$2\,a^{\frac{1}{2}} \cdot 5\,a^{\frac{3}{2}} =$$

28. (91) Find the following quotient:

$$\frac{8\,x}{12} \div \frac{9\,x^3}{15} =$$

29. (92) Using the facts in the following table showing church contributions, fill in the blanks below the table:

INDIVIDUAL CONTRIBUTIONS	NUMBER OF CONTRIBUTORS
0	2
1 cent	23
5 cents	42
10 cents	13
25 cents	9
$50 check	1

The arithmetic average is______________________________
The mode is______________________________
The median is ______________________________

30. (93) Write an algebraic expression which will be a translation of the following statement: "15° plus $\frac{1}{2}$ of the complement of an angle."

*Answer*__________

APPENDIX

I. TABLES

1. NUMBER OF CORRECT SOLUTIONS

Table I, on page 102, gives the total number of correct answers to the exercises or problems in the various tests.

As explained on page 50, tests 22 and 23, on logarithms and the slide rule, were not used in the scales, and are therefore omitted from the table. The numbers in the first column show the original order of the exercises.

2. ORDER OF DIFFICULTY OF THE EXERCISES

In Table II, on page 103, the first column of numbers at the left indicates the final order of the exercises when arranged according to their increasing difficulty as shown by the percentage of pupils solving them. The figures in the various columns at the right give the original order of the exercises and problems.

For example, in test 1 (see page 32) the sixth exercise in the final order was the tenth in the original order. By reference to Table I, on page 102, it is seen that this exercise was solved correctly by 1075 pupils, while the sixth exercise in the original list was solved correctly by only 1053 pupils.

3. PERCENTAGES OF "RIGHTS," P. E. VALUES, AND SCALE VALUES

Table III, which begins on page 104, shows the following items for the different tests listed in the table on page 102:

1. The original order of the exercises and problems.
2. The percentage of the entire group that solved each exercise or problem.
3. The P. E. equivalents of the difference in each group between 50 per cent and the percentage of pupils solving each exercise or problem correctly.
4. The scale values which correspond to the P. E. values.

TABLE I

Number of Problem	NUMBER OF TEST																				
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	24
1	1174	1097	765	732	533	346	959	418	636	277	530	388	372	464	673	242	62	25	58	660	129	230
2	1171	1113	907	908	528	215	937	466	596	332	525	688	465	392	648	194	34	13	53	576	146	294
3	1152	1093	661	897	550	321	738	610	621	496	481	629	465	446	462	300	92	33	31	649	26	253
4	1125	729	941	869	367	363	306	530	632	449	430	604	423	453	629	214	147	75	40	395	135	270
5	1097	679	908	770	753	434	291	511	300	541	343	669	120	348	639	292	100	101	54	340	124	264
6	1053	885	903	777	646	296	583	504	506	356	544	637	446	144	641	310	97	93	24	238	92	267
7	888	892	821	774	761	296	491	689	466	308	309	609	406	316	416	239	103	97		476	20	268
8	963	891	558	883	392	448	505	369	489	538	311	325	287	322	555	201	38	86		312	61	269
9	642	445	945	733	597	369	604	521	517	536	262	475	478	221	256	314	77	56		355	64	146
10	1075	910	891	730	553	300	520		531	518	311	530	385	139	554	208	63	28		269		225
11	963	963	544	680	610	311	448		501	505	222	229	363	303	560	218	91	41		457		204
12	769	609	888	762	551	345			507	525	248	506			405	231	87					35
13	564	708	736	521	712	349			409	510	122	213			363	222						34
14	996	595	874	814	651	104			146	164		267			307	271						220
15	921	771	283	786	500	174			39	468		476			277	223						219
16	963	323	665	568	595	366			56	412		616			445							207
17	851	258	450	527	604	390			43	460		463			132							200
18	781		783	485	623	358			96	417		339			365							70
19	774		799	597	613	315				371		606			342							135
20	646		523	527	588	153				287		355			453							9
21	788		747	368	407	390				292		554										113
22	796		617	353		123				349		501										135
23	538		693							392		445										
24	812		663							355		128										
25	771		886									50										
26	545		847									56										
27	477		822																			
28	397		833																			
29	665		461																			
30	195																					
31	439																					

TABLE II

Final Order of Exercises	Number of Test																					
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	24
1	1	2	9	2	7	8	1	7	1	5	6	2	9	1	1	9	4	5	1	1	2	2
2	2	1	4	3	5	5	2	3	4	8	1	5	2	4	2	6	7	7	5	3	4	4
3	3	3	5	8	13	17	3	4	3	9	2	6	3	3	6	3	5	6	2	2	1	8
4	4	11	2	4	14	21	9	9	2	12	3	3	6	2	5	5	6	8	4	6	5	7
5	5	10	6	14	6	9	6	5	10	10	4	16	4	5	4	14	3	4	3	7	6	6
6	10	7	10	15	18	16	10	6	9	13	5	21	7	8	11	7	11	9	6	11	9	5
7	6	8	12	6	19	4	8	2	6	11	8	7	10	7	8	1	12	11		4	8	3
8	14	6	14	7	11	18	7	1	12	3	10	19	1	11	10	12	9	3		9	3	1
9	8	15	25	5	17	1	11	8	11	15	7	4	11	9	18	15	10	10		5	7	10
10	11	4	26	12	9	12	4		8	17	12	22	8	6	3	13	1	1		8		14
11	16	13	28	9	16	13	5		7	4	9	10	5	10	20	11	8	2		10		15
12	15	5	27	1	20	3			13	18	11	12			16	4	2					16
13	7	12	7	10	10	19			5	16	13	23			7	10						11
14	17	14	19	11	12	11			14	23		15			12	8						17
15	24	9	18	16	3	10			18	19		9			13	2						9
16	22	16	1	17	1	7			16	1		17			19							19
17	21	17	21	20	2	6			17	6		20			14							22
18	18		13	13	15	2			15	22		1			15							21
19	19		23	18	21	15				24		18			9							18
20	25		16	19	8	20				2		8			17							12
21	12		24	22	4	22				7		24										13
22	29		3	21		14				20		14										20
23	20		22							21		11										
24	9		8							14		13										
25	13		11									26										
26	26		20									25										
27	23		29																			
28	27		17																			
29	31		15																			
30	28																					
31	30																					

TABLE III

TEST 1

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	97.5	– 2.905	51
(2)	97.3	– 2.857	52
(3)	95.7	– 2.546	55
(4)	93.4	– 2.234	58
(5)	91.1	– 1.997	61
(6)	87.5	– 1.706	63
(7)	73.8	– 0.945	71
(8)	80.0	– 1.248	68
(9)	53.3	– 0.123	79
(10)	89.3	– 1.843	62
(11)	80.0	– 1.248	68
(12)	63.9	– 0.527	75
(13)	46.9	+ 0.115	81
(14)	82.8	– 1.397	66
(15)	76.5	– 1.071	70
(16)	80.0	– 1.248	68
(17)	70.7	– 0.807	72
(18)	64.9	– 0.567	75
(19)	64.3	– 0.543	75
(20)	53.7	– 0.138	79
(21)	65.4	– 0.588	74
(22)	66.1	– 0.616	74
(23)	44.7	+ 0.198	82
(24)	67.4	– 0.669	74
(25)	64.0	– 0.531	75
(26)	45.3	+ 0.175	81
(27)	39.6	+ 0.391	83
(28)	33.0	+ 0.652	86
(29)	55.2	– 0.194	78
(30)	16.2	+ 1.462	94
(31)	36.5	+ 0.512	85

TEST 2

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	91.7	– 2.054	59
(2)	93.0	– 2.199	58
(3)	91.4	– 2.026	60
(4)	61.0	– .414	76
(5)	56.8	– 0.254	77
(6)	74.0	– 0.954	71
(7)	74.6	– 0.982	70
(8)	75.0	– 0.977	70
(9)	37.2	+ 0.484	84
(10)	76.1	– 1.052	70
(11)	80.5	– 1.275	67
(12)	50.9	– 0.033	80
(13)	59.2	– 0.345	76
(14)	49.8	+ 0.007	80
(15)	64.5	– 0.551	75
(16)	27.0	+ 0.909	89
(17)	21.6	+ 1.165	91

TEST 3

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	79.1	− 1.201	69	(16)	68.8	− 0.727	73
(2)	93.8	− 2.281	57	(17)	46.5	+ 0.130	81
(3)	68.4	− 0.710	73	(18)	81.0	− 1.302	67
(4)	97.3	− 2.857	52	(19)	82.6	− 1.391	66
(5)	93.9	− 2.293	57	(20)	54.0	− 0.149	79
(6)	93.4	− 2.234	58	(21)	77.2	− 1.105	69
(7)	84.9	− 1.531	65	(22)	63.8	− 0.523	75
(8)	57.7	− 0.288	77	(23)	71.7	− 0.851	72
(9)	97.7	− 2.958	51	(24)	68.6	− 0.719	73
(10)	92.1	− 2.093	59	(25)	91.6	− 2.044	60
(11)	56.3	− 0.235	78	(26)	87.6	− 1.713	63
(12)	91.8	− 2.064	60	(27)	85.0	− 1.537	65
(13)	76.1	− 1.052	70	(28)	86.1	− 1.609	64
(14)	90.9	− 1.979	61	(29)	47.7	+ 0.085	81
(15)	29.3	+ 0.807	87				

TEST 4

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	73.8	− 0.945	71	(12)	76.8	− 1.086	69
(2)	91.5	− 2.035	60	(13)	52.5	− 0.093	80
(3)	90.4	− 1.935	61	(14)	82.1	− 1.363	67
(4)	87.6	− 1.713	63	(15)	79.2	− 1.206	68
(5)	77.6	− 1.125	69	(16)	57.3	− 0.273	78
(6)	78.3	− 1.160	69	(17)	53.1	− 0.115	79
(7)	78.0	− 1.145	69	(18)	68.9	+ 0.041	80
(8)	89.0	− 1.819	62	(19)	40.0	+ 0.376	83
(9)	73.9	− 0.949	71	(20)	53.1	− 0.115	79
(10)	73.6	− 0.936	71	(21)	37.1	+ 0.489	86
(11)	68.5	− 0.714	73	(22)	35.6	+ 0.547	85

TEST 5

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	61.8	– 0.445	76	(12)	82.5	– 1.386	66
(2)	61.2	– 0.422	76	(13)	63.8	– 0.523	75
(3)	63.7	– 0.519	75	(14)	75.4	– 1.019	70
(4)	42.5	+ 0.280	82	(15)	57.9	– 0.296	77
(5)	87.3	– 1.692	64	(16)	68.9	– 0.713	73
(6)	74.9	– 0.996	70	(17)	70.0	– 0.778	73
(7)	88.2	– 1.757	63	(18)	72.2	– 0.873	72
(8)	45.4	+ 0.172	81	(19)	71.0	– 0.820	72
(9)	69.2	– 0.744	73	(20)	68.1	– 0.698	74
(10)	64.1	– 0.535	75	(21)	47.2	+ 0.104	81
(11)	70.7	– 0.807	72				

TEST 6

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	73.7	– 0.913	71	(12)	72.9	– 0.904	71
(2)	45.5	– 0.168	81	(13)	73.8	– 0.945	71
(3)	67.9	– 0.689	73	(14)	22.0	+ 1.145	90
(4)	76.7	– 1.081	70	(15)	36.8	+ 0.500	84
(5)	91.8	– 2.064	59	(16)	77.4	– 1.115	69
(6)	58.4	– 0.315	77	(17)	82.5	– 1.386	65
(7)	62.6	– 0.476	76	(18)	75.7	+ 1.033	70
(8)	94.7	– 2.397	57	(19)	66.6	– 0.636	74
(9)	78.0	– 1.145	69	(20)	32.3	+ 0.681	87
(10)	63.4	– 0.508	75	(21)	82.5	– 1.386	65
(11)	65.8	– 0.603	74	(22)	26.0	+ 0.954	88

TEST 7

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	95.9	– 2.579	55	(7)	49.1	+ 0.033	80
(2)	93.7	– 2.269	58	(8)	50.5	– 0.019	80
(3)	73.8	– 0.945	70	(9)	60.4	– 0.391	76
(4)	30.6	+ 0.752	88	(10)	52.0	– 0.074	79
(5)	29.1	+ 0.816	88	(11)	44.8	+ 0.194	82
(6)	58.3	– 0.311	77				

TEST 8

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	60.1	– 0.380	77	(6)	66.2	– 0.620	74
(2)	61.2	– 0.422	76	(7)	90.5	– 1.944	61
(3)	80.2	– 1.259	67	(8)	48.5	+ 0.056	79
(4)	69.6	– 0.761	73	(9)	68.5	– 0.714	73
(5)	67.1	– 0.656	74				

TEST 9

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	89.1	– 1.827	62	(10)	74.4	– 0.972	71
(2)	80.0	– 1.248	68	(11)	70.2	– 0.786	73
(3)	87.0	– 1.670	64	(12)	71.0	– 0.820	72
(4)	88.5	– 1.780	63	(13)	57.3	– 0.273	78
(5)	42.0	+ 0.299	83	(14)	40.1	+ 0.372	84
(6)	70.9	– 0.816	74	(15)	15.8	+ 1.487	95
(7)	65.3	– 0.583	75	(16)	22.7	+ 1.110	91
(8)	68.5	– 0.714	73	(17)	17.4	+ 1.391	93
(9)	72.4	– 0.882	71	(18)	38.9	+ 0.418	84

TEST 10

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	63.5	– 0.512	75	(13)	89.3	– 1.843	62
(2)	58.1	– 0.303	77	(14)	28.7	+ 0.834	88
(3)	86.9	– 1.663	64	(15)	82.0	– 1.357	67
(4)	78.6	– 1.176	69	(16)	72.2	– 0.873	72
(5)	94.7	– 2.397	56	(17)	80.6	– 1.279	68
(6)	62.3	– 0.464	76	(18)	73.0	– 0.909	71
(7)	53.9	– 0.145	79	(19)	65.0	– 0.571	75
(8)	94.2	– 2.331	57	(20)	50.3	– 0.011	80
(9)	93.9	– 2.293	57	(21)	51.1	– 0.041	80
(10)	90.7	– 1.962	61	(22)	61.1	– 0.418	76
(11)	88.4	– 1.772	63	(23)	68.7	– 0.723	73
(12)	91.9	– 2.074	60	(24)	62.2	– 0.461	76

TEST 11

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	93.0	− 2.188	58	(8)	54.6	− 0.172	79
(2)	92.1	− 2.093	59	(9)	46.0	+ 0.149	81
(3)	84.4	− 1.499	66	(10)	54.6	− 0.172	79
(4)	75.4	− 1.019	70	(11)	38.9	+ 0.418	83
(5)	60.2	− 0.383	78	(12)	50.1	− 0.004	80
(6)	95.4	− 2.498	55	(13)	21.4	+ 1.176	91
(7)	54.2	− 0.156	79				

TEST 12

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	47.4	+ 0.097	80	(14)	37.4	+ 0.476	85
(2)	96.5	− 2.686	54	(15)	66.8	− 0.644	74
(3)	88.2	− 1.757	63	(16)	86.4	− 1.629	64
(4)	84.7	− 1.518	65	(17)	64.9	− 0.567	75
(5)	93.8	− 2.281	59	(18)	47.5	+ 0.093	81
(6)	89.3	− 1.843	62	(19)	85.0	− 1.537	65
(7)	85.4	− 1.563	65	(20)	54.8	− 0.179	79
(8)	45.6	+ 0.164	82	(21)	85.5	− 1.569	65
(9)	66.6	− 0.636	74	(22)	77.3	− 1.110	69
(10)	74.3	− 0.968	71	(23)	68.7	− 0.723	73
(11)	32.1	+ 0.689	86	(24)	38.6	+ 0.430	84
(12)	71.0	− 0.820	72	(25)	15.1	+ 1.531	95
(13)	29.9	+ 0.782	87	(26)	16.9	+ 1.421	94

TEST 13

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	53.7	− 0.138	79	(7)	58.6	− 0.322	77
(2)	67.1	− 0.656	73	(8)	41.4	+ 0.322	83
(3)	67.1	− 0.656	73	(9)	69.0	− 0.735	73
(4)	61.0	− 0.414	76	(10)	55.6	− 0.209	78
(5)	17.3	+ 1.397	93	(11)	52.4	− 0.089	80
(6)	64.4	− 0.547	75				

TEST 14

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	94.5	− 2.370	58	(7)	64.4	− 0.547	74
(2)	79.8	− 1.238	68	(8)	65.6	− 0.596	74
(3)	90.8	− 1.971	61	(9)	45.0	+ 0.187	80
(4)	92.3	− 2.114	59	(10)	28.3	+ 0.851	89
(5)	70.9	− 0.816	72	(11)	61.7	− 0.441	76
(6)	29.3	+ 0.807	88				

TEST 15

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	97.1	− 2.811	52	(11)	80.8	− 1.291	67
(2)	93.5	− 2.245	58	(12)	58.4	− 0.315	77
(3)	66.7	− 0.640	74	(13)	52.4	− 0.089	79
(4)	90.8	− 1.971	61	(14)	44.3	+ 0.213	82
(5)	92.2	− 2.103	59	(15)	40.0	+ 0.376	83
(6)	92.5	− 2.134	59	(16)	64.2	− 0.539	75
(7)	60.0	− 0.376	77	(17)	19.0	+ 1.302	93
(8)	80.1	− 1.253	68	(18)	52.7	− 1.000	70
(9)	36.9	+ 0.496	85	(19)	49.4	+ 0.022	80
(10)	79.9	− 1.243	68	(20)	65.4	− 0.588	74

TEST 16

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	68.0	− 0.693	74	(9)	88.2	− 1.757	62
(2)	54.5	− 0.168	79	(10)	58.4	− 0.315	77
(3)	84.3	− 1.493	65	(11)	61.2	− 0.422	76
(4)	59.0	− 0.337	77	(12)	64.9	− 0.567	74
(5)	82.0	− 1.357	66	(13)	62.4	− 0.468	76
(6)	87.1	− 1.677	64	(14)	76.1	− 1.052	69
(7)	67.1	− 0.656	73	(15)	62.6	− 0.476	75
(8)	56.5	− 0.243	78				

TEST 17

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	21.9	+ 1.150	91	(7)	36.4	+ 0.516	85
(2)	12.0	+ 1.742	97	(8)	13.4	+ 1.643	96
(3)	32.5	+ 0.673	86	(9)	27.2	+ 0.900	89
(4)	51.9	− 0.071	80	(10)	22.3	+ 1.130	91
(5)	35.3	+ 0.559	85	(11)	32.2	+ 0.685	87
(6)	34.3	+ 0.600	86	(12)	30.7	+ 0.748	88

TEST 18

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	23.4	+ 1.076	90	(7)	90.7	– 1.962	61
(2)	12.1	+ 1.735	97	(8)	80.4	– 1.269	67
(3)	30.8	+ 0.744	87	(9)	52.3	– 0.085	80
(4)	70.1	– 0.782	72	(10)	26.2	+ 0.945	89
(5)	94.4	– 2.357	57	(11)	38.3	+ 0.441	84
(6)	86.9	– 1.663	63				

TEST 19

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	54.7	– 0.175	79	(4)	37.7	+ 0.464	84
(2)	50.0	0.000	80	(5)	50.9	– 0.033	80
(3)	29.2	+ 0.812	88	(6)	22.6	+ 1.115	91

TEST 20

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	90.0	– 1.900	61	(7)	64.9	– 0.567	75
(2)	78.6	– 1.176	69	(8)	42.6	+ 0.277	82
(3)	88.5	– 1.780	63	(9)	48.4	+ 0.059	80
(4)	53.9	– 0.145	79	(10)	36.7	+ 0.504	85
(5)	46.4	+ 0.134	81	(11)	62.3	– 0.464	76
(6)	32.5	– 0.673	74				

TEST 21

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	62.0	– 0.453	76	(6)	44.2	+ 0.216	82
(2)	70.1	– 0.782	73	(7)	09.6	+ 1.935	99
(3)	22.4	+ 1.125	91	(8)	29.3	+ 0.807	88
(4)	64.9	– 0.567	75	(9)	30.8	+ 0.744	87
(5)	59.6	– 0.360	77				

TEST 24

Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value	Original Order of Exercises	Percentage of Rights	P.E. Value	Scale Value
(1)	70.3	− 0.790	72	(12)	10.7	+ 1.843	98
(2)	89.9	− 1.892	62	(13)	10.4	+ 1.867	98
(3)	77.4	− 1.115	69	(14)	67.3	− 0.665	74
(4)	82.6	− 1.391	66	(15)	67.0	− 0.652	74
(5)	80.7	− 1.286	68	(16)	63.3	− 0.504	75
(6)	81.7	− 1.340	67	(17)	61.2	− 0.422	76
(7)	82.0	− 1.357	67	(18)	21.4	+ 1.176	91
(8)	82.3	− 1.374	67	(19)	41.3	+ 0.326	83
(9)	44.6	+ 0.201	82	(20)	03.1	+ 2.769	107
(10)	68.8	− 0.727	73	(21)	34.6	+ 0.588	86
(11)	62.4	− 0.468	76	(22)	41.3	− 0.326	83

4. PROBLEMS OMITTED BECAUSE THEY WERE TOO EASY

The problems listed below were the easiest ones for the pupils to solve in the composite test given at the end of the year's work. They were not included in the composite scales.

1. (54) Construct an angle of 56° on the working line AB by using only the protractor.

A———————————————B

2. (56) Solve the following equation:

$$8\,x + 30 - 4\,x = 60 + 3\,x.$$

3. (56) By using the compasses show the sum of the given lines a and b on the working line below.

a

b

4. (56) Complete the following statement:

A straight angle is an angle________________________

5. (57) Write but do not solve the equation which expresses the condition in the following problem:

If four times a number increased by 6 is 42, find the number.

6. (58) Express by an equation the area (A) of a rectangle a inches long and b inches wide.

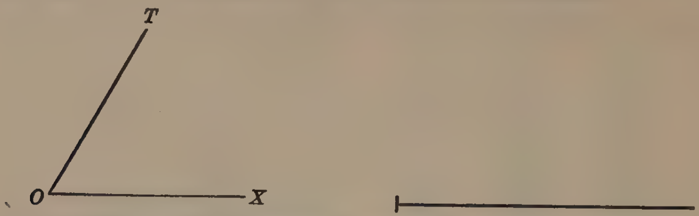
7. (59) Perform the operations indicated in the following:

$$15\,y - y + 4\,y =$$

8. (60) Subtract as indicated in the following:

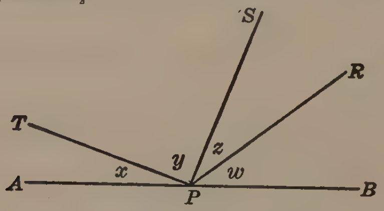
$$14\,x - 3\,x =$$

9. (60) On the working line below construct an angle equal to the given angle XOT by using only the ruler and compasses.



10. (60) Express by an equation the area (A) of a rectangle 8 in. long and 5 in. wide.

11. (61) Express the sums of the angles in the accompanying figure by means of an equation.

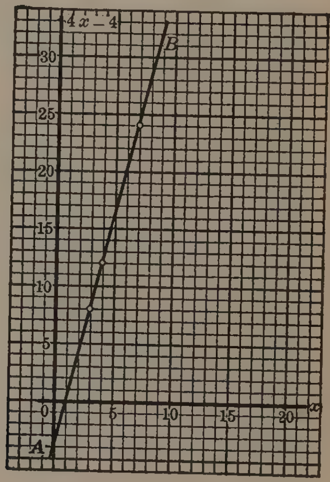


12. (61) Perform the operations indicated below and simplify the result:

$$3(5x - 1) =$$

13. (61) The accompanying diagram represents a graph of the function $4x - 4$. Pick out from the graph the values of x that will make the function $4x - 4$ have the following values:

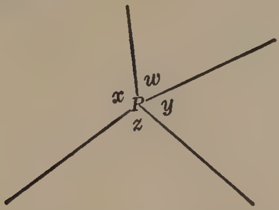
1. If $4x - 4 = 0$, then $x = 0$.
2. If $4x - 4 = 12$, then $x = 4$.
3. If $4x - 4 = 28$, then $x = 8$.



14. (63) Complete the following statement:

An acute angle is an angle________________________

15. (63) Express the sum of the angles in the accompanying figure by means of an equation.



16. (63) Perform the operations indicated below and simplify the result:

$$7(2x + 4) =$$

5. PROBLEMS OMITTED BECAUSE THEY WERE TOO DIFFICULT

The problems listed below were the hardest ones for the pupils to solve in the composite test given at the end of the year's work. They were not included in the composite scales.

1. (94) Perform the operations indicated in the following expression:

$$\frac{2a}{3} + \frac{5a}{8} - \frac{a}{4} =$$

2. (94) Write but do not solve the equation which expresses the condition in the following problem:

A can do a piece of work in 3 da. and B can do it in 4 da. How long will it take the two men to do the work if they work together?

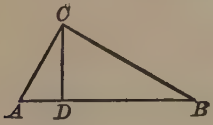
3. (94) If y varies inversely with x, and $y = 8$ when $x = 12$, find y when $x = 4$.

4. (95) Perform the indicated operations and reduce to the simplest form in the following expression:

$$\frac{2\sqrt{5c} \cdot \sqrt{12c}}{3c\sqrt{15}} =$$

5. (96) Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

6. (97) In the triangle ABC, CD is perpendicular to AB and angle $C = 90°$. Show below that triangle ACD is similar to triangle BCD.



7. (99) Write but do not solve the equation which expresses the conditions of the following problem:

A train leaves Chicago at the rate of 50 mi. an hour. Three hours later another train leaves the same station and travels over the same road in the same direction at the rate of 60 mi. an hour. How far from Chicago will the second train overtake the first?

8. (99) Given that $C = \frac{C}{R + r}$, solve for r.

9. (99) Perform the operations indicated below and reduce the result to the simplest form:

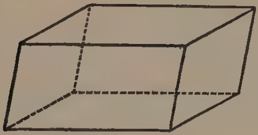
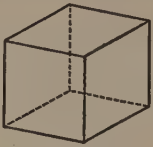
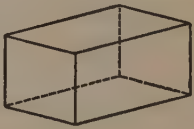
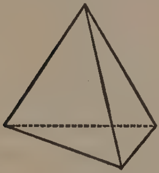
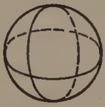
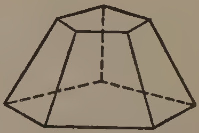
$$\sqrt{x^4 + 2x^3 + 7x^2 + 6x + 9} =$$

10. (100) Write but do not solve the equation which expresses the conditions of the following problem:

A boy had 7 times as many pennies as his sister. After he had given 9 pennies to his sister he still had 36 more pennies than she. How many pennies had each at first?

11. (101) Given that $S = \frac{a}{1 - r}$, solve for r.

12. (102) Write the correct names below each of the following:

13. (103) Given that $L = \frac{MX - g}{X}$, solve for M.

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